



STUDY GUIDE

MATH STUDIES

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# IB Academy Mathematics Studies Study Guide

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Version: MatSt.1.0.170410



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# INTRODUCTION

Welcome to the IB.Academy Study Guide for IB Mathematics Studies.

We are proud to present our study guides and hope that you will find them helpful. They are the result of a collaborative undertaking between our tutors, students and teachers from schools across the globe. Our mission is to create the most simple yet comprehensive guides accessible to IB students and teachers worldwide. We are firm believers in the open education movement, which advocates for transparency and accessibility of academic material. As a result, we embarked on this journey to create these study guides that will be continuously reviewed and improved. Should you have any comments, feel free to contact us.

For this Mathematics Studies guide, we incorporated everything you need to know for your final exam. The guide is broken down into chapters based on the syllabus topics and they begin with 'cheat sheets' that summarise the content. This will prove especially useful when you work on the exercises. The guide then looks into the subtopics for each chapter, followed by our step-by-step approach and a calculator section which explains how to use the instrument for your exam.

For more information and details on our revision courses, be sure to visit our website at [ib.academy](http://ib.academy). We hope that you will enjoy our guides and best of luck with your studies.

IB.Academy Team



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# BASICS

## 1.1 Notation

To begin with, it is crucial to understand some mathematical terminology that you will hear over and over again as you work through your IB math exam. Questions might ask you to ‘set up an equation’ or an ‘inequality’, so it is important that you know what this means.



**Equation** contains an “=” sign.

$$\begin{aligned} \text{e.g. } -2x - 3 &= 5 \\ -2x &= 8 \\ x &= -4 \end{aligned}$$

**Inequality** contains a  $>$ ,  $<$ ,  $\geq$  or  $\leq$  sign.

e.g.  $-2x - 3 \geq 5$  ( $-2x - 3$  is greater than or equal to 5).  
Solve like an equation, except if you  $\times$  or  $\div$  by a negative number, then reverse the inequality!

$$\begin{aligned} -2x - 3 &\geq 5 \\ -2x &\geq 8 \\ x &\leq -4 \end{aligned}$$

$0 < a < 1$  means:  $a$  is between 0 and 1 (not including 0 and 1)

**Absolute value**  $|x|$  is the positive version of  $x$  (distance from 0).

$$\begin{aligned} \text{e.g. } |3| &= 3 \\ |-3| &= 3 \end{aligned}$$

$1 \leq |x| \leq 2$  means:  $x$  is between 1 and 2 or between  $-2$  and  $-1$ .

## 1.2 Laws of exponents

Exponents always follow certain rules. If you are multiplying or dividing, use the following rules to determine what happens with the powers.

Example.

$x^1 = x$	$6^1 = 6$
$x^0 = 1$	$7^0 = 1$
$x^m \cdot x^n = x^{m+n}$	$4^5 \cdot 4^6 = 4^{11}$
$\frac{x^m}{x^n} = x^{m-n}$	$\frac{3^5}{3^4} = 3^{5-4} = 3^1 = 3$
$(x^m)^n = x^{m \cdot n}$	$(10^5)^2 = 10^{10}$
$(x \cdot y)^n = x^n \cdot y^n$	$(2 \cdot 4)^3 = 2^3 \cdot 4^3$ and $(3x)^4 = 3^4 x^4$
$x^{-1} = \frac{1}{x}$	$5^{-1} = \frac{1}{5}$ and $\left(\frac{3}{4}\right)^{-1} = \frac{4}{3}$
$x^{-n} = \frac{1}{x^n}$	$3^{-5} = \frac{1}{3^5} = \frac{1}{243}$

## 1.3 Unit conversion

Units are used to measure different kinds of factors in the world; for example temperature, weight or price are all things that can be measured in different units. Measured values can however only be compared if they are in the same unit; so while you may know the price of one object in EUR and of another in USD, in order to determine which one is more expensive, you will need to convert the price of both objects into one currency. Therefore particularly when applying mathematics to real world problems, you will often need to convert between units.



**SI units** are the base units from which other units are derived. The 7 base units are: meter, kilogram, second, ampere, kelvin, mole, candela.

e.g. the 'meter' is the SI unit used to measure distance; other units used to measure distance like the centimeter (0.01 meters) or the kilometer (100 meters) are based on the meter.



# NUMBERS AND ALGEBRA

## Table of contents & cheatsheet

### 2.1. Estimation

10

#### Rounding to a

- decimal place
- number of significant figures

**Error** approximate value – exact value =  $V_A - V_E$

**Percentage error**  $\left| \frac{V_A - V_E}{V_E} \right| \times 100$

**Standard form**  $a \times 10^k$ , where  $1 \leq |a| < 10$  and  $k \in \mathbb{Z}$

### Compound interest

$$FV = PV \times \left(1 + \frac{r}{100k}\right)^{kn}$$

Where

FV	Future Value
PV	Present Value
$r$	rate (%)
$k$	compounding frequency
$n$	overall length of time

### 2.2. Sequences and series

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**Arithmetic:** +/– common difference

$$u_n = n^{\text{th}} \text{ term} = u_1 + (n-1)d$$

$$S_n = \text{sum of } n \text{ terms} = \frac{n}{2}(2u_1 + (n-1)d)$$

with  $u_1 = a = 1^{\text{st}}$  term,  $d =$  common difference.

**Geometric:**  $\times/\div$  common ratio

$$u_n = n^{\text{th}} \text{ term} = u_1 \cdot r^{n-1}$$

$$S_n = \text{sum of } n \text{ terms} = \frac{u_1(1-r^n)}{(1-r)}$$

$$S_\infty = \text{sum to infinity} = \frac{u_1}{1-r}, \text{ when } -1 < r < 1$$

with  $u_1 = a = 1^{\text{st}}$  term,  $r =$  common ratio.

### Exponents

$$x^1 = x$$

$$x^0 = 1$$

$$x^m \cdot x^n = x^{m+n}$$

$$\frac{x^m}{x^n} = x^{m-n}$$

$$(x^m)^n = x^{m \cdot n}$$

$$(x \cdot y)^n = x^n \cdot y^n$$

$$x^{-1} = \frac{1}{x}$$

$$x^{-n} = \frac{1}{x^n}$$

## 2.1 Estimation

### 2.1.1 Rounding

In math you come across rounding almost all the time, so its important to know how to do it accurately. The key things you need to know are:

1. which number you should be rounding;
2. whether you should round up or down.

You can round any number using these two questions:

#### What does the rounded digit become?

- If the digit is  $< 5$ , it stays the same.
- If the digit is  $\geq 5$ , add +1 to the digit

e.g. 201.78095

*Round to the nearest 10 and 10,000<sup>th</sup>*

*Nearest 10*

Look at the next digit  $\rightarrow 1$

$1 < 5 \Rightarrow 200$

*Nearest 10,000<sup>th</sup> (= 0.0001)*

Look at the next digit  $\rightarrow 5$

$5 \geq 5 \Rightarrow$  add +1 to 9 which carries over,  
 $\Rightarrow 201.7810$ .

#### Which digit is being rounded? (2 possibilities)

- A certain decimal place  
 e.g. 201.78095 rounded to:
  - $\rightarrow$  2 decimal places  $\Rightarrow 201.78$
  - $\rightarrow$  1 decimal place  $\Rightarrow 201.8$
- A certain number of significant figures

**Rule:** zeros to the left of the first non-zero digit are not significant

**All other:** numbers are significant

e.g. 0.0023045 rounded to:

- $\rightarrow$  2 significant figures  $\Rightarrow 0.0023$
- $\rightarrow$  3 significant figures  $\Rightarrow 0.00230$
- $\rightarrow$  4 significant figures  $\Rightarrow 0.002305$

The word "estimate" without further detail means "estimate to 1 significant figure". When nothing is specified, always round to 3 significant figures.

## 2.1.2 Errors



**The error** tells you by how much an estimate differed from the actual value.

This can be done by calculating the approximate value — exact value

$$V_A - V_E$$

**Percentage error**  $\left| \frac{\text{approximate value} - \text{exact value}}{\text{exact value}} \right| \times 100$

$$\left| \frac{V_A - V_E}{V_E} \right| \times 100$$

Example.

John estimates a 119.423 cm piece of plywood to be 100 cm. What is the error?

$$\begin{aligned} \text{Error} &= V_A - V_E \\ &= 100 - 119.423 \\ &= -19.423 \approx -19.4 \end{aligned}$$

What is the percentage error?

$$\begin{aligned} \text{Percentage error} &= \left| \frac{100 - 119.423}{119.423} \right| \times 100 \\ &= |-0.1626| \times 100 \\ &= 0.1626 \times 100 \approx 16.3\% \end{aligned}$$

## 2.1.3 Standard form



**Standard form** is just a way of rewriting any number, sometimes also referred to as ‘scientific notation’. This should be in the form  $a \times 10^k$ , where  $a$  is between 1 and 10, and  $k$  is an integer.

10	1 × 10 <sup>1</sup>
1000	1 × 10 <sup>3</sup>
3280	3.28 × 10 <sup>3</sup>
4582000	4.582 × 10 <sup>6</sup>

## 2.2 Sequences and series

### 2.2.1 Arithmetic sequence



**Arithmetic sequence** the next term is the previous number + the common difference ( $d$ ).

e.g.  $2, 4, 6, 8, 10, \dots$   $d = +2$  and  $2, -3, -8, -13, \dots$   $d = -5$

To find the common difference  $d$ , subtract two consecutive terms of an arithmetic sequence from the term that follows it, i.e.  $u_{(n+1)} - u_n$ .

**DB 1.1** Use the following equations to calculate the  $n^{\text{th}}$  term or the sum of  $n$  terms.

$$u_n = u_1 + (n - 1)d \qquad S_n = \frac{n}{2}(2u_1 + (n - 1)d)$$

with

$$u_1 = a = 1^{\text{st}} \text{ term} \qquad d = \text{common difference}$$

Often the IB requires you to first find the  $1^{\text{st}}$  term and/or common difference.

#### Finding the first term $u_1$ and the common difference $d$ from other terms.

In an arithmetic sequence  $u_{10} = 37$  and  $u_{22} = 1$ . Find the common difference and the first term.

- |    |  |  |
|----|--|--|
| 1. | Put numbers in to $n^{\text{th}}$ term formula | $37 = u_1 + 9d$<br>$1 = u_1 + 21d$             |
| 2. | Equate formulas to find $d$                    | $21d - 1 = 9d - 37$<br>$12d = -36$<br>$d = -3$ |
| 3. | Use $d$ to find $u_1$                          | $1 - 21 \cdot (-3) = u_1$<br>$u_1 = 64$        |

## 2.2.2 Geometric sequence



**Geometric sequence** the next term is the previous number multiplied by the common ratio ( $r$ ).

To find the common ratio, divide any term of an arithmetic sequence by the term that precedes it, i.e.  $\frac{\text{second term } (u_2)}{\text{first term } (u_1)}$  e.g.  $2, 4, 8, 16, 32, \dots$   $r = 2$

and  $25, 5, 1, 0.2, \dots$   $r = \frac{1}{5}$

Use the following equations to calculate the  $n^{\text{th}}$  term, the sum of  $n$  terms or the sum to infinity when  $-1 < r < 1$ .

DB 1.1

$$\begin{array}{lll}
 u_n = n^{\text{th}} \text{ term} & S_n = \text{sum of } n \text{ terms} & S_\infty = \text{sum to infinity} \\
 = u_1 \cdot r^{n-1} & = \frac{u_1(1-r^n)}{(1-r)} & = \frac{u_1}{1-r}
 \end{array}$$

again with

$$u_1 = a = 1^{\text{st}} \text{ term} \qquad r = \text{common ratio}$$

Similar to questions on Arithmetic sequences, you are often required to find the 1<sup>st</sup> term and/or common ratio first.

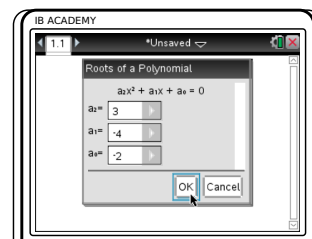
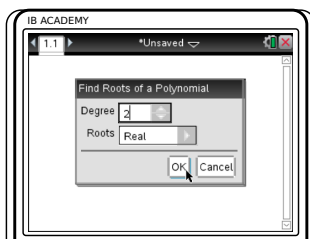
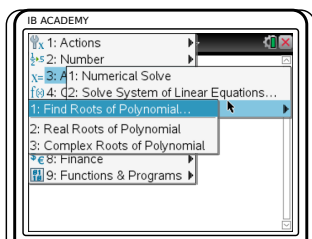
## 2.3 GDC solvers (TI-Nspire)


There are several handy tools on your GDC which will help you answer most of the more complicated algebra questions. You can use these in cases where you are looking to find the roots of a quadratic equation or solve a pair of simultaneous equations.


### APPS → PIYSMLT2

#### Solving quadratic equations

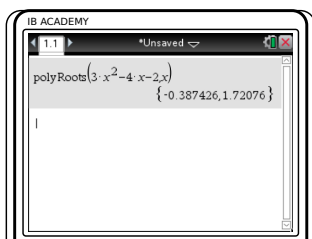
Solve  $3x^2 - 4x - 2 = 0$



Press , choose  
 3: Algebra  
 3: Polynomial Tools  
 1: Find Roots of Polynomial

Degree = 2, Roots = Real,  
 press 

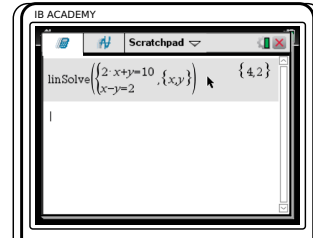
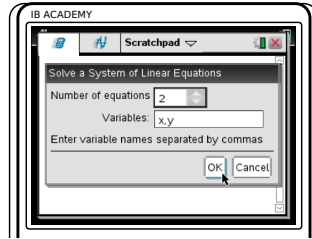
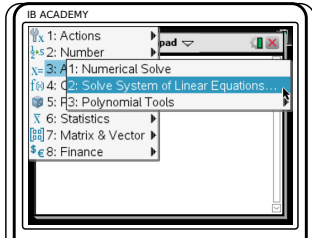
Enter values a2, a1 and a0.  
 Press 




so  $x = 1.72$  or  $x = -0.387$

### Solving Simultaneous equations

$2x + y = 10$  and  $x - y = 2$ ; find the values of  $x$  and  $y$




Press , choose  
3: Algebra  
2: Solve System of  
Linear Equations

Number of equations = 2.

Press 

Enter the two equations

press 

So  $x = 4$  and  $y = 2$

### APPS → FINANCE → TVM SOLVER

You can also use your GDC for questions dealing with money and interest rates. The TVM Solver (“Time Value of Money”) allows you to fill in all the variables you know and solve for the missing one.

$$FV = PV \times \left(1 + \frac{r}{100k}\right)^{kn}$$

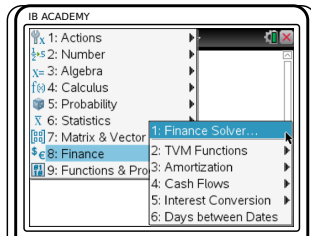
For some questions you might find it simpler to use the formula for compound interest in your data booklet!


Table 2.1: Abbreviations

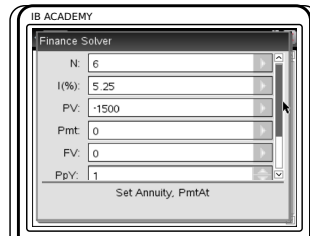
Abbreviation	Stands for
TVM	Time Value of Money
N	Number of years
I%	percentage Interest rate
PV	Present Value - should be negative
PMT	PayMenT
FV	Future Value
P/Y	Payments per Year
C/Y	Compounding periods per Year

## Solving questions about compound interest

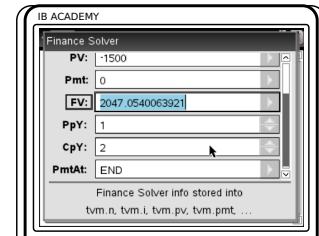
\$1500 is invested at 5.25% per annum. The interest is compounded twice per year. How much will it be worth after 6 years?

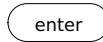


Press , choose  
8: Finance  
1: Finance Solver



Enter all known values  
For this example:  
N=6 (years)  
I=5.25 (interest rate)  
PV=-1500 (present value)  
negative because  
investment represents cash  
outflow;  
PMT=0  
FV=0 (future value)  
P/Y=1 (payment/yr)  
C/Y=2 (compound/yr)



Highlight cell of asked  
value, in this case FV,  
press 

So FV = \$2047.05



# DESCRIPTIVE STATISTICS

## Table of contents & cheatsheet

### Definitions

**Population** the entire group from which statistical data is drawn (and which the statistics obtained represent).

**Sample** the observations actually selected from the population for a statistical test.

**Random Sample** a sample that is selected from the population with no bias or criteria; the observations are made at random.

**Discrete** finite or countable number of possible values. (e.g. money, number of people)

**Continuous** infinite amount of increments. (e.g. time, weight)

Note: continuous data can be presented as discrete data, e.g. if you round time to the nearest minute or weight to the nearest kilogram.

### 3.1. Descriptive statistics

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**Mean** the average value,

$$\bar{x} = \frac{\text{the sum of the data}}{\text{no. of data points}}$$

**Mode** the value that occurs most often

**Median** when the data set is ordered low to high and the number of data points is:

- odd: the median is the middle value;
- even: the median is the average of the two middle values.

**Range** largest  $x$ -value – smallest  $x$ -value

**Variance**  $\sigma^2 = \frac{\sum f(x - \bar{x}^2)}{n}$  *calculator only*

**Standard deviation**  $\sigma = \sqrt{\text{variance}}$  *calculator only*

**Grouped data:** data presented as an interval. Use the midpoint as the  $x$ -value in all calculations.

$Q_1$  first quartile = 25<sup>th</sup> percentile.

$Q_2$  median = 50<sup>th</sup> percentile

$Q_3$  third quartile = 75<sup>th</sup> percentile

$Q_3 - Q_1$  interquartile range (IQR) = middle 50 percent

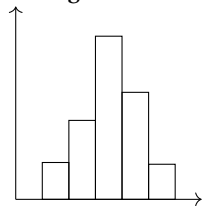
### 3.2. Statistical graphs

20

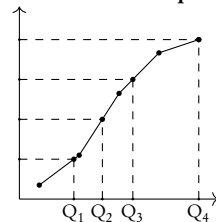
**Frequency** the number of times an event occurs in an experiment

**Cumulative frequency** the sum of the frequency for a particular class and the frequencies for all the the classes below it

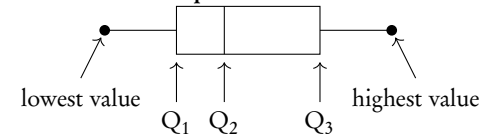
**Histogram**



**Cumulative frequency**



**Box and whisker plot**



### 3.1 Descriptive statistics

The mean, mode and median, are all ways of measuring “averages”. Depending on the distribution of the data, the values for the mean, mode and median can differ slightly or a lot. Therefore, the mean, mode and median are all useful for understanding your data set.



**Example data set:** 6, 3, 6, 13, 7, 7 in a table:

$x$	3	6	7	13
<b>frequency</b>	1	2	2	1

**Mean** the average value,  $\bar{x} = \frac{\text{the sum of the data}}{\text{no. of data points}} = \frac{\sum x}{n} = \frac{\sum fx}{\sum f}$   
 e.g.  $\bar{x} = \frac{3+6+6+7+7+13}{6} = \frac{1 \cdot 3 + 2 \cdot 6 + 2 \cdot 7 + 1 \cdot 13}{1+2+2+1} = 7$

**Mode** the value that occurs most often (highest frequency)  
 e.g. The example data set has 2 modes: 6 and 7

**Median** the middle value when the data set is ordered low to high. Even number of values: the median is the average of the two middle values.  
 Find for larger values as  $n + \frac{1}{2}$ .  
 e.g. data set from low to high: 3, 6, 6, 7, 7, 13  
 median =  $\frac{6+7}{2} = 6.5$

**Range** largest  $x$ -value — smallest  $x$ -value  
 e.g. range =  $13 - 3 = 10$

**Variance**  $\sigma^2 = \frac{\sum f(x - \bar{x})^2}{n}$  *calculator only*

**Standard deviation**  $\sigma = \sqrt{\text{variance}}$  *calculator only*

**Note on grouped data:** data presented as an interval; e.g. 10–20 cm.

- Use the midpoint as the  $x$ -value in all calculations. So for 10–20 cm use 15 cm.
- For 10–20 cm, 10 is the lower boundary, 20 is the upper boundary and the width is  $20 - 10 = 10$ .

Adding a constant to all the values in a data set or multiplying the entire data set by a constant influences the mean and standard deviation values in the following way:

Table 3.1: Adding or multiplying by a constant

	adding constant $k$	multiplying by $k$
mean	$\bar{x} + k$	$k \times \bar{x}$
standard deviation	$\sigma$	$k \times \sigma$



$Q_1$	first quartile	= 25 <sup>th</sup> percentile. The value for $x$ so that 25% of all the data values are $\leq$ to it.
$Q_2$	median	= 50 <sup>th</sup> percentile
$Q_3$	third quartile	= 75 <sup>th</sup> percentile
$Q_3 - Q_1$	interquartile range (IQR)	= middle 50 percent

Example.

Snow depth is measured in centimeters:  
30, 75, 125, 55, 60, 75, 65, 65, 45, 120, 70, 110.

Find the range, the median, the lower quartile, the upper quartile and the interquartile range.

First always rearrange data into ascending order: 30, 45, 55, 60, 65, 65, 70, 75, 75, 110, 120, 125

- The range:

$$125 - 30 = 95 \text{ cm}$$

- The median: there are 12 values so the median is between the 6<sup>th</sup> and 7<sup>th</sup> value.

$$\frac{65 + 70}{2} = 67.5 \text{ cm}$$

- The lower quartile: there are 12 values so the lower quartile is between the 3<sup>rd</sup> and 4<sup>th</sup> value.

$$\frac{55 + 60}{2} = 57.5 \text{ cm}$$

- The upper quartile: there are 12 values so the lower quartile is between the 9<sup>th</sup> and 10<sup>th</sup> value.

$$\frac{75 + 110}{2} = 92.5 \text{ cm}$$

- The IQR

$$92.5 - 57.5 = 35 \text{ cm}$$

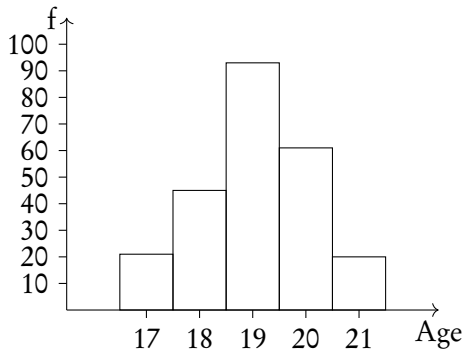
### 3.2 Statistical graphs



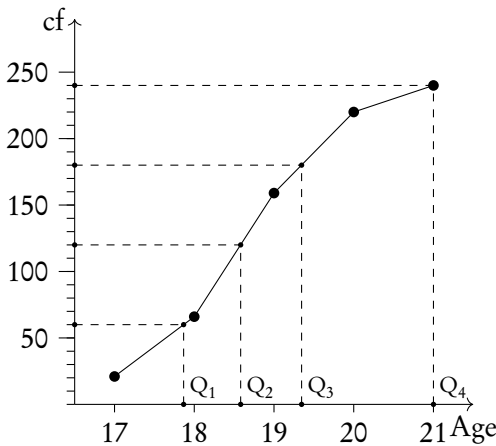
**Frequency** the number of times an event occurs in an experiment

**Cumulative frequency** the sum of the frequency for a particular class and the frequencies for all the classes below it

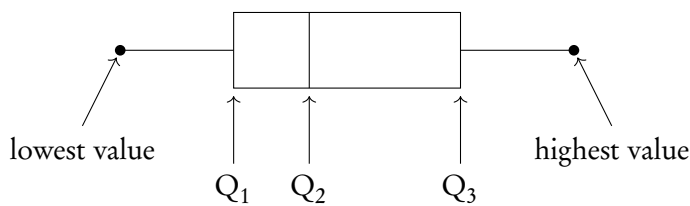
Age	17	18	19	20	21
No. of students	21	45	93	61	20
Cumulative freq.	21	66	159	220	240



A histogram is used to display the frequency for a specific condition. The frequencies (here: # of students) are displayed on the  $y$ -axis, and the different classes of the sample (here: age) are displayed on the  $x$ -axis. As such, the differences in frequency between the different classes assumed in the sample can easily be compared.



The cumulative frequency graph is used to display the development of the frequencies as the classes of the event increase. The graph is plotted by using the sum of all frequencies for a particular class, added to the frequencies for all the classes below it. The classes of the event (age) are displayed on the  $x$ -axis, and the frequency is displayed on the  $y$ -axis. The cumulative frequency graph always goes upwards, because the cumulative frequency increases as you include more classes.



Box and whisker plots neatly summarize the distribution of the data. It gives information about the range, the median and the quartiles of the data. The first and third quartiles are at the ends of the box, the median is indicated with a vertical line in the interior of the box, and the maximum and minimum points are at the ends of the whiskers.

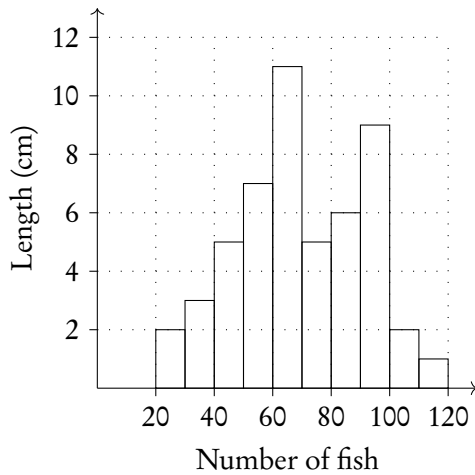


**Outliers** will be any points lower than  $Q_1 - 1.5 \times IQR$  and larger than  $Q_3 + 1.5 \times IQR$  (IQR = interquartile range)

To identify the value of  $Q_1$ ,  $Q_2$  and  $Q_3$ , it is easiest to use the cumulative frequency graph. First, determine the percentage of the quartile in question. Second, divide the total cumulative frequency of the graph (i.e. the total sample size) by 100 and multiply by the corresponding percentage. Then, you will have found the frequency ( $y$ -value) at which 25% for  $Q_1$  / 50% for  $Q_2$  / 75% for  $Q_3$  of the sample is represented. To find the  $x$ -value, find the corresponding  $x$ -value for the previously identified  $y$ -value.

Example.

Using the histogram, create a cumulative frequency graph and use it to construct a box and whisker diagram.

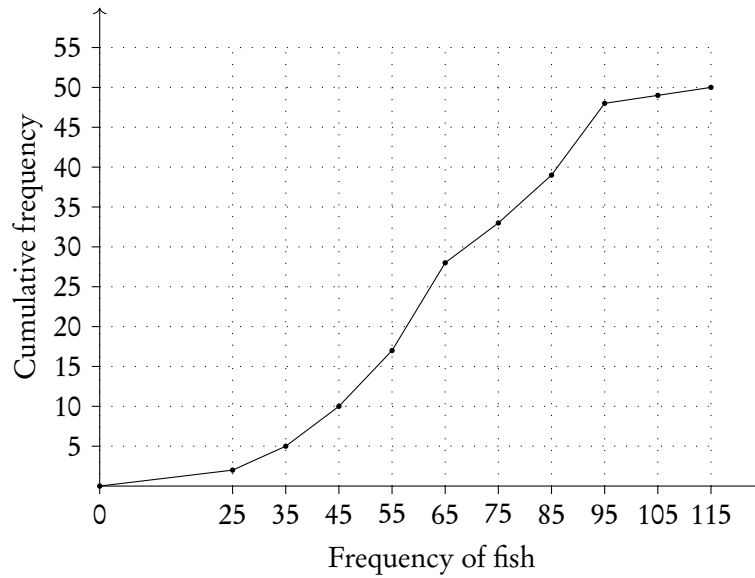


Write out the table for frequency and cumulative frequency.

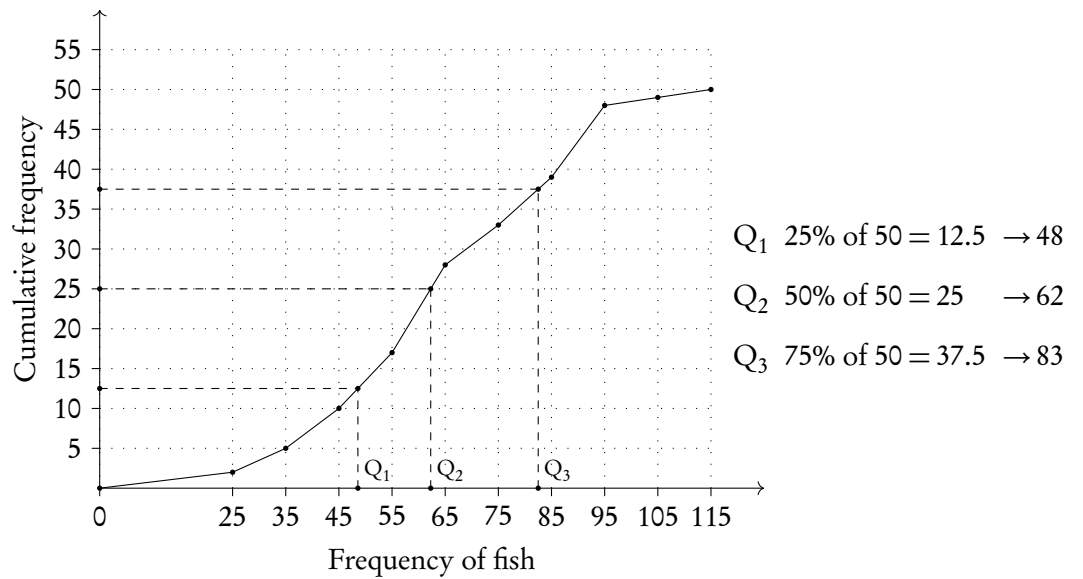
Frequency of fish	20-30	30-40	40-50	50-60	60-70	70-80	80-90	90-100	100-110	110-120
Length of fish	2	3	5	7	11	5	6	9	1	1
Cumulative f.	2	5	10	17	28	33	39	48	49	50

Example.

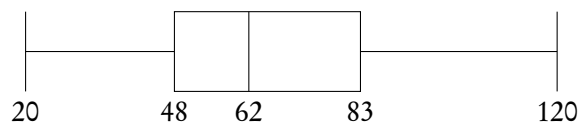
Plot on cumulative frequency chart. Remember to use the midpoint of the date, e.g., 25 for 20–30.



Use graph to find  $Q_1$ ,  $Q_2$  and  $Q_3$ .



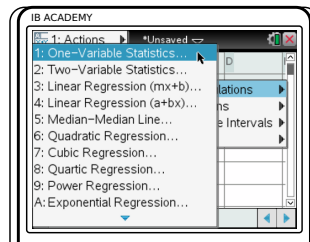
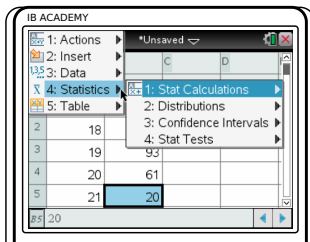
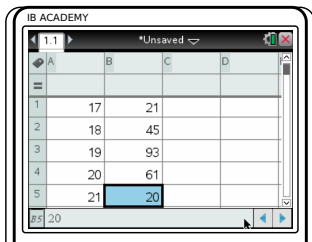
Plot box and whiskers.



### 3.3 GDC (TI-Nspire)

**To find mean, standard deviation and quartiles etc.**

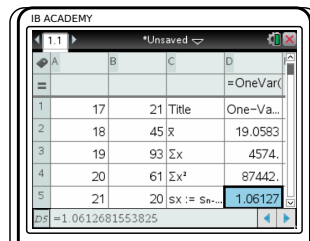
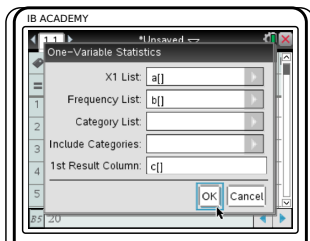
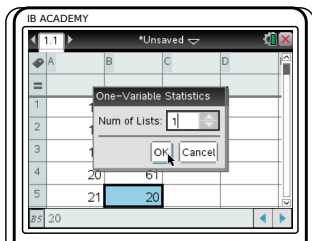
For the data used in the previous example showing the ages of students



Press  $\text{on}$ , go to Lists and Spreadsheets. Enter  $x$ -values in L1 and, if applicable, frequencies in L2

Press  $\text{menu}$ , choose 4: Statistics  
1: Stat Calculations

1: One-Variable Statistics



Enter Num of lists: 1.  
Press  $\text{OK}$

Enter names of columns you used to enter your  $x$ -list and frequency list and column where you would like the solutions to appear: a[], b[] and c[].  
Press  $\text{OK}$

mean = 19.06;  
standard deviation = 1.06 etc.





# LOGIC, SETS AND PROBABILITY

## Table of contents & cheatsheet

### 4.1. Logic

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**A proposition** is any statement that can be either true or false, mathematical or not.

Propositions		Negation		Conjunction	Disjunction	Exclusive disjunction	Implication
$p$	$q$	$\neg p$	$\neg q$	$p \wedge q$	$q \vee p$	$p \nabla q$	$q \Rightarrow p$
T	T	F	F	T	T	F	T
T	F	F	T	F	T	T	F
F	T	T	F	F	T	T	T
F	F	T	T	F	F	F	T

**Implication** ( $p \Rightarrow q$ ) if  $p$ , then  $q$ .

Other types of implication: **Converse** ( $q \Rightarrow p$ ), **Inverse** ( $\neg p \Rightarrow \neg q$ ), **Contrapositive** ( $\neg q \Rightarrow \neg p$ ).

**Equivalence** ( $p \Leftrightarrow q$ )  $p$  if and only if  $q$ .

**Tautology** a statement that is always true.

**Contradiction** a statement that is always false.

### 4.2. Sets

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**Set** any collection of things with a common property (capital letter, curly brackets)

e.g.  $A = \{2, 4, 6, 8\}$

**Number of elements in a set**  $n(A) = 4$

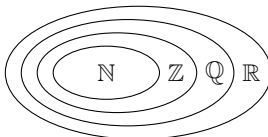
**A member of a set**  $6 \in A$

**An empty set**  $\emptyset$

**Subset** a set contained in another set.

e.g.  $B = \{4, 8\} \Rightarrow B \subset A$

Sets can be shown in Venn diagrams.



Natural numbers ( $\mathbb{N}$ )

Integers ( $\mathbb{Z}$ )

Rational numbers ( $\mathbb{Q}$ )

Real numbers ( $\mathbb{R}$ )

### 4.3. Probability

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**Sample space** the list of all possible outcomes.

**Event** the outcomes that meet the requirement.

**Probability** for event  $A$ ,

$$P(A) = \frac{\text{Number of ways } A \text{ can happen}}{\text{all outcomes in the sample space}}$$

**Conditional probability** used for successive events that come after one another. The probability of  $A$ , given that  $B$  has happened:  $P(A|B) = \frac{P(A \cap B)}{P(B)}$ .

#### Probability distributions

A fair coin is tossed twice.

	H	T
H	HH	HT
T	TH	TT

Table of probability distribution ( $x$  is the number of heads obtained)

$x$	0	1	2
$P(X = x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

The sum of  $P(X = x) = 1$ .

**Expected value of  $X$**   $E(X) = \sum xP(X = x) =$

$$= 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} = 1$$

## 4.1 Logic

### 4.1.1 Propositions



A **proposition** is any statement that can be either true or false, mathematical or not.

2 propositions		Negation	Conjunction	Disjunction	Exclusive disjunction
Bob studies	Bob drinks	Bob does <b>not</b> study	Bob studies <b>and</b> drinks	Bob studies <b>or</b> drinks <b>or both</b>	Bob studies <b>or</b> drinks <b>not both</b>
$p$	$q$	$\neg p$	$p \wedge q$	$p \vee q$	$p \underline{\vee} q$
T	T	F	T	T	F
T	F	F	F	T	T
F	T	T	F	T	T
F	F	T	F	F	F

### 4.1.2 Implications

$p \Rightarrow q$  is only false when  $p = T$  and  $q = F$ .

If  $p$ , then  $q$  ( $p \Rightarrow q$ )  
 e.g.  $p$ : “you steal”,  $q$ : “you go to prison”.

**Implication:**  
 If you steal, then you go to prison.

$p$	$q$	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Contrapositive and implication are equivalent and converse and inverse are equivalent.

Propositions	Negation	Implications	Converse	Inverse	Contrapositive
$x = 16$	$x$ is a square number $x \neq 16$	$x$ is <b>not</b> a square number If $x = 16$ then $x$ is a square number	If $x$ is a square number then $x = 16$	If $x \neq 16$ then $x$ is <b>not</b> a square number	If $x$ is <b>not</b> a square number then $x \neq 16$
$p$	$q$	$\neg p$	$\neg q$	$p \Rightarrow q$	$q \Rightarrow p$
$\neg p \Rightarrow \neg q$	$\neg q \Rightarrow \neg p$	$T$	$T$	$T$	$T$
$T$	$T$	$F$	$F$	$T$	$T$
$T$	$F$	$F$	$T$	$F$	$F$
$F$	$T$	$T$	$F$	$T$	$T$
$F$	$F$	$T$	$T$	$T$	$T$

### 4.1.3 Equivalence ( $p \Leftrightarrow q$ )

An equivalence has an identical implication ( $p \Rightarrow q$ ) and converse ( $q \Rightarrow p$ ), i.e.  $p$  if and only if  $q$ .

e.g.  $p$ :  $x$  is even,  $q$ :  $x$  is a multiple of 2.



**Tautology** a statement that is always true

$$(p \Rightarrow q) \Leftrightarrow (\neg q \Rightarrow \neg p)$$

**Contradiction** a statement that is always false

$$\neg(p \Rightarrow q) \Leftrightarrow (\neg q \Rightarrow \neg p)$$

## 4.2 Sets



**Set** any collection of things with a common property (capital letter, curly brackets)

$$\begin{aligned} \text{e.g. } A &= \{2, 4, 6, 8\} \\ &= \{\text{even numbers between 1 and 9}\} \\ &= \{x \mid x \text{ is even, } 1 < x < 9\} \end{aligned}$$

**Number of elements in a set**  $n(A) = 4$

**A member of a set**  $6 \in A$

**An empty set**  $\emptyset$

**Subset** a set contained in another set.

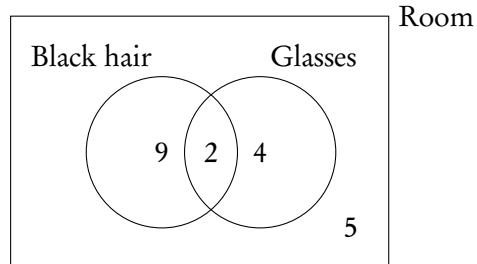
$$\begin{aligned} \text{e.g. } B &= \{\text{multiplies of 4 between 0 and 8}\} \\ B &= \{4, 8\} \Rightarrow B \subset A \end{aligned}$$

### 4.2.1 Venn diagrams

Sets can be shown using.

Example.

A room of 20 people, 11 have black hair, 6 have glasses, 2 have both.



\* When drawing venn diagrams, start from the middle.

### 4.2.2 Number sets



**Natural numbers**  $\mathbb{N} = 0, 1, 2, 3 \dots$

**Integers**  $\mathbb{Z} = \dots -3, -2, -1, 0, 1, 2, 3, \dots$

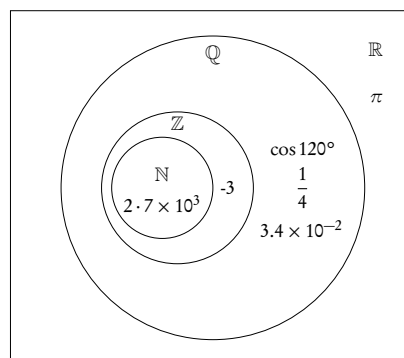
**Rational numbers**  $\mathbb{Q}$ ; all integers and fractions

**Real numbers**  $\mathbb{R}$ ; all rational and irrational numbers ( $\pi, \sqrt{2}$ , etc.)

Example.

Place the following numbers on the Venn diagram:

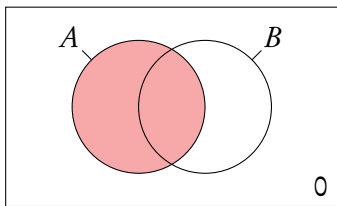
$\frac{1}{4}, -3, \pi, \cos 120^\circ, 2.7 \times 10^3, 3.4 \times 10^{-2}$



## 4.3 Probability

### 4.3.1 Single events (Venn diagrams)

Probability for single events can be expressed through venn diagrams.



**Sample space** the list of all possible outcomes.

**Event** the outcomes that meet the requirement.

**Probability** for event  $A$ ,

$$P(A) = \frac{\text{Number of ways } A \text{ can happen}}{\text{all outcomes in the sample space}}$$

Here the shaded circle.

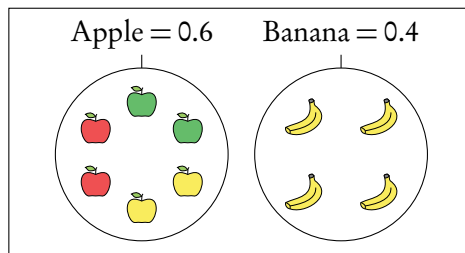
Imagine I have a fruit bowl containing 6 apples and 4 bananas.



Example.

I pick a piece of fruit.

What is the probability of picking each fruit?

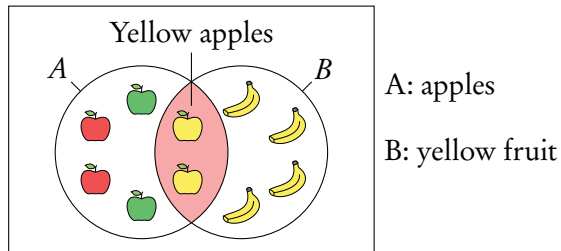


As apples cannot be bananas this is mutually exclusive, therefore  $P(A \cup B) = P(A) + P(B)$  and  $P(A \cap B) = 0$ . It is also an exhaustive event as there is no other options apart from apples and bananas. If I bought some oranges the same diagram would then be not exhaustive (oranges will lie in the sample space).

In independent events  $P(A \cap B) = P(A) \times P(B)$ . It will often be stated in questions if events are independent.

Example.

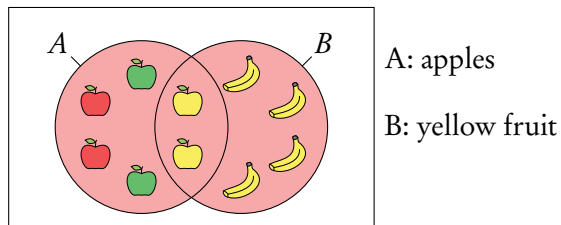
Of the apples 2 are red, 2 are green and 2 are yellow.  
 What is the probability of picking a yellow apple?



This is not mutually exclusive as both apples and bananas are yellow fruits. Here we are interested in the intersect  $P(A \cap B)$  of apples and yellow fruit, as a yellow apple is in both sets  $P(A \cap B) = P(A) + P(B) - P(A \cup B)$ .

Example.

What is the probability of picking an apple or a yellow fruit?



When an event is exhaustive the probability of the union is 1.

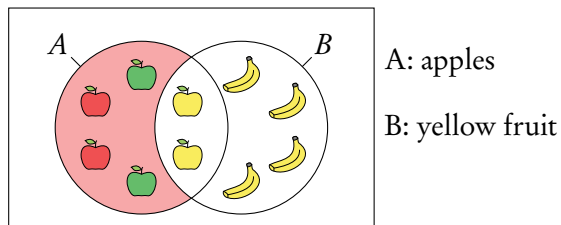
This is a union of two sets: apple and yellow fruit.

The union of events  $A$  and  $B$  is:

- when  $A$  happens;
- when  $B$  happens;
- when both  $A$  and  $B$  happen  $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ .

Example.

What is the probability of not picking a yellow fruit?

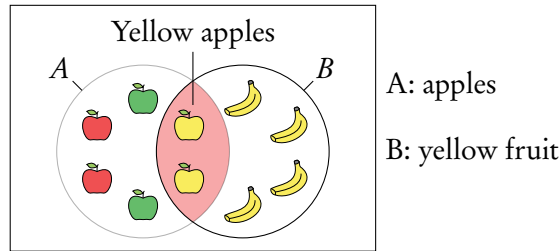


This is known as the complement of  $B$  or  $B'$ .  $B' = 1 - B$ .

Here we are interested in everything but the yellow fruit.

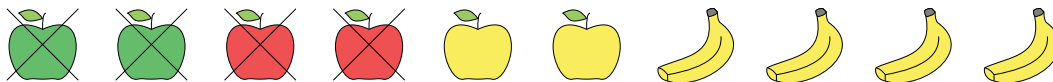
Example.

What is the probability of picking an apple given I pick a yellow fruit?



This is “conditional” probability in a single event. **Do not use the formula in the formula booklet.** Here we are effectively narrowing the sample space =  $\frac{0.2}{(0.2 + 0.4)} = \frac{1}{3}$ .

You can think of it like removing the non yellow apples from the fruit bowl before choosing.



Conditional probability  $P(A|B) = \frac{P(A \cap B)}{P(B)}$ .

### 4.3.2 Multiple events (tree Diagrams)



**Dependent events** two events are dependent if the outcome of event *A* affects the outcome of event *B* so that the probability is changed.

**Independent events** two events are independent if the fact that *A* occurs does not affect the probability of *B* occurring.

**Conditional probability** used for successive events that come one after another (as in tree diagrams). The probability of *A*, given that *B* has happened:  $P(A|B) = \frac{P(A \cap B)}{P(B)}$ .

Questions involving dependent events will often involve elements that are drawn “without replacement”. Remember that the probabilities will be changing with each new set of branches.

Probabilities for successive events can be expressed through tree diagrams. In general, if you are dealing with a question that asks for the probability of:

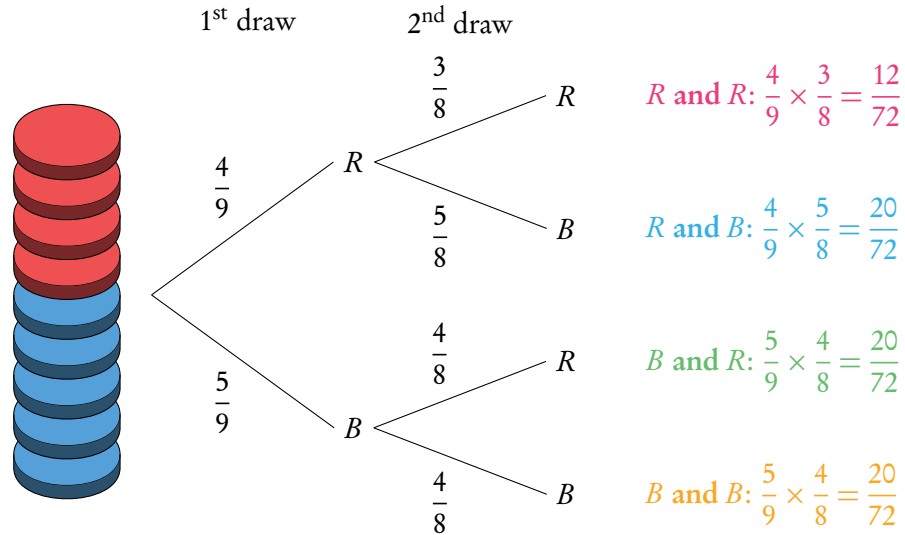
- one event **and** another, you **multiply**
- one event **or** another, you **add**

Example.

Two disks are randomly drawn without replacement from a stack of 4 red and 5 blue disks. Draw a tree diagram for all outcomes.

The probability of drawing two red disks can be found by multiplying both probabilities of getting red  $\left(\frac{4}{9} \times \frac{3}{8}\right)$ .

The probabilities for each event should always add up to 1. The probabilities describing all the possible outcomes should also equal 1 (that is, the probabilities that we found by multiplying along the individual branches).



What is the probability to draw one red and one blue disk?

$P(\text{one red and one blue})$

$$\begin{aligned} & (P(R) \text{ and } P(B)) \quad \text{or} \quad (P(B) \text{ and } P(R)) \\ & (P(R) \times P(B)) \quad \quad \quad (P(B) \times P(R)) \\ & \frac{20}{72} \quad \quad \quad + \quad \quad \quad \frac{20}{72} \quad \quad \quad = \frac{40}{72} = \frac{5}{9} \end{aligned}$$

It is common for conditional probability questions to relate to previous answers.

What is the probability to draw at least one red disk?

$P(\text{at least one red})$

$$\begin{aligned} P(R \text{ and } R) + P(B \text{ and } R) + P(R \text{ and } B) &= 1 - P(B \text{ and } B) \\ \frac{12}{72} + \frac{20}{72} + \frac{20}{72} &= 1 - \frac{20}{72} = \frac{52}{72} = \frac{13}{18} \end{aligned}$$

What is the probability of picking a blue disc given that at least one red disk is picked?

$$P(\text{blue disk} \mid \text{at least one red disk}) = \frac{P(\text{a blue disk})}{P(\text{at least one red disk})} = \frac{\frac{5}{9}}{\frac{13}{18}} = \frac{10}{13}$$



Another way of dealing with multiple events is with a sample space diagram or a probability distribution.

**Probability distributions.**

A fair coin is tossed twice,  $X$  is the number of heads obtained.

1. Draw a sample space diagram

	H	T
H	H, H	H, T
T	T, H	T, T

2. Tabulate the probability distribution

$x$	0	1	2
$P(X = x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

(The sum of  $P(X = x)$  always equals 1)

3. Find the expected value of  $X$ :  $E(X)$

$$E(X) = \sum xP(X = x)$$

$$= 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} = 1$$

So if you toss a coin twice, you expect to get heads once.

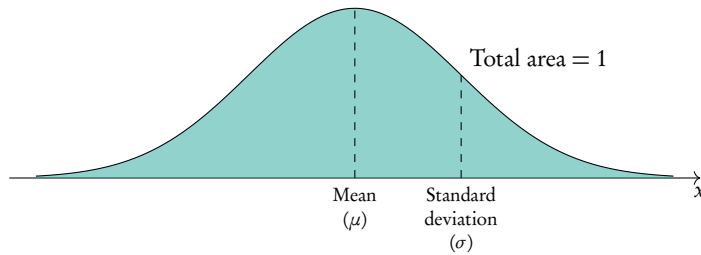


# STATISTICAL APPLICATIONS

## Table of contents & cheatsheet

### 5.1. Normal distribution

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Use normalcdf: for the probability that  $x$  is between any 2 values.

Use invnorm: to get an  $x$ -value for a given probability.

**Expected value** the value of  $x$  multiplied by probability  $E(x) = x \cdot p$ .

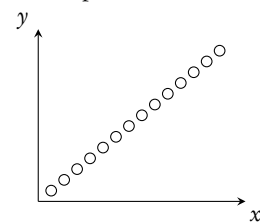
### 5.2. Bivariate statistics

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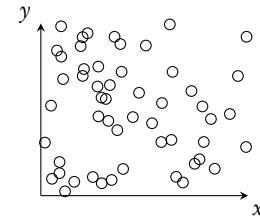
Using data where two variables ( $x, y$ ) are measured.

#### Scatter diagrams

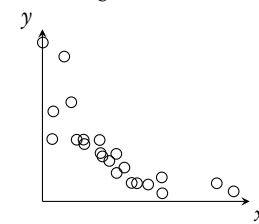
Perfect positive



No correlation



Weak negative



*Correlation does not mean causation.*

Pearson's correlation  $-1 \leq r \leq 1$

Interpretation of  $r$ -values

$r$ -value	very weak	weak	moderate	strong
correlation	$0.00 \leq  r  \leq 0.25$	$0.25 \leq  r  \leq 0.50$	$0.50 \leq  r  \leq 0.75$	$0.75 \leq  r  \leq 1.00$

**Regression equation** a mathematical model would best describe the relationship between the two measured variables; when drawn manually, always passes through the mean point  $(\bar{x}, \bar{y})$ .

### 5.3. Chi-squared test

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**Chi-square test** Used to test independence of two variables. Using  $\chi^2$  value and/or  $p$ -value.

$H_0$  the variables are independent (*null hypothesis*)

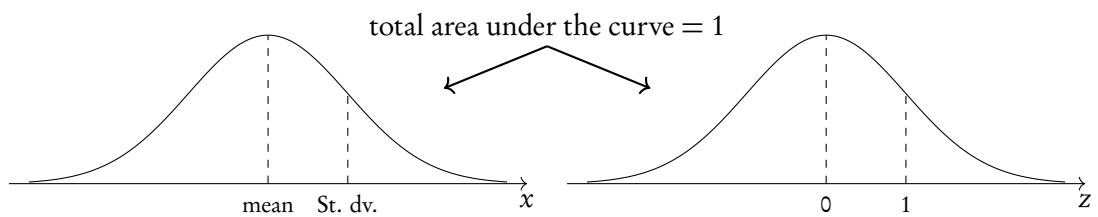
$H_1$  the variables are not independent (*alternative hypothesis*)

If critical value  $< \chi^2$  or  $p$ -value  $<$  significant level (for 10% test, significant level = 0.1) *reject null hypothesis*.

## 5.1 Normal distribution

A normal distribution is one type of probability distribution which gives a bell-shape curve if all the values and their corresponding probabilities are plotted.

We can use normal distributions to find the probability of obtaining a certain value or a range of values. This can be found using the area under the curve; the area under the bell-curve between two  $x$ -values always corresponds to the probability for getting an  $x$ -value in this range. The total area under the normal distribution is always 1; this is because the total probability of getting any  $x$ -value adds up to 1 (or, in other words, you are 100% certain that your  $x$ -value will lie somewhere on the  $x$ -axis below the bell-curve).



### 5.1.1 Using GDC

You can use your GDC to work through questions dealing with normal distributions. In these questions you will either need to find probabilities for given  $x$ -values or  $x$ -values for given probabilities. In both cases, you will need to know the mean ( $\mu$ ) and standard deviation ( $\sigma$ ) for the given example.

Note: even though you will be using your GDC to find probabilities for normal distributions, it's always very useful to draw a diagram to indicate for yourself (and the examiner) what area or  $x$ -value you are looking for.

Use `normal cdf (lowerbound, upperbound,  $\mu$ ,  $\sigma$ )`: for the probability that  $x$  is between any 2 values.

- For lower bound =  $-\infty$ , use `-1E99`
- For upper bound =  $\infty$ , use `1E99`

Use `invnorm ( $\rho$ ,  $\mu$ ,  $\sigma$ )`: to get an  $x$ -value for a given probability.

\*Note: the calculator assumes  $\rho$  is to the *left* of  $x$ . When  $\rho$  is to the right of  $x$ , subtract the output from 1 to get the final answer.



**Expected value** the value of  $x$  multiplied by probability.

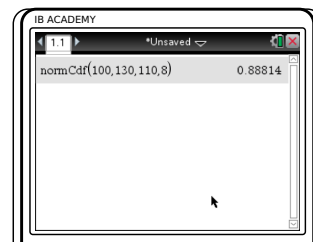
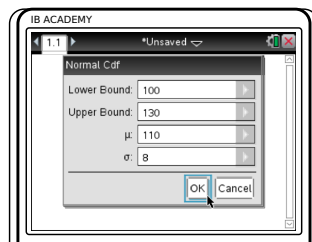
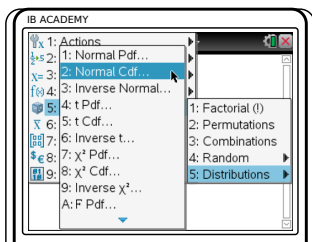
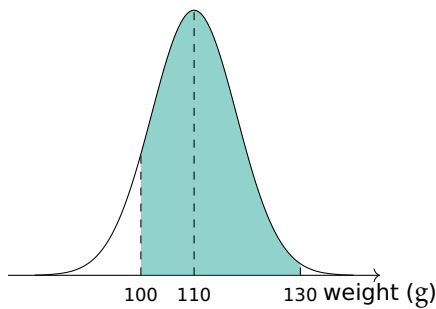
**To find a probability or percentage of a whole (the area under a normal distribution curve)**


The weights of pears are normally distributed with mean = 110 g and standard deviation = 8 g.  
Find the percentage of pears that weigh between 100 g and 130 g

Sketch!

Indicate:

- The mean = 110 g
- Lower bound = 100 g
- Upper bound = 130 g
- And shade the area you are looking to find.



Press , choose  
5: Probability  
5: Distributions  
2: Normal Cdf

Enter lower and upper boundaries, mean ( $\mu$ ) and standard deviation ( $\sigma$ ).  
For lower bound =  $-\infty$ ,  
set lower: -1E99  
For upper bound =  $\infty$ ,  
set upper: 1E99

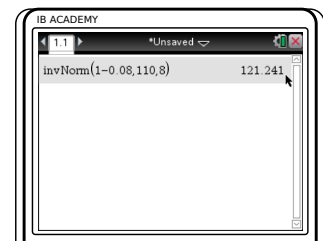
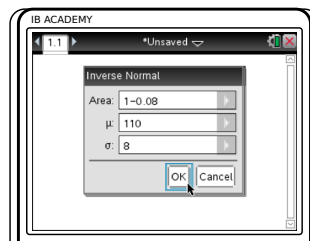
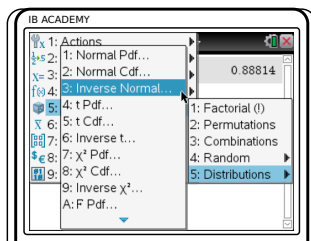
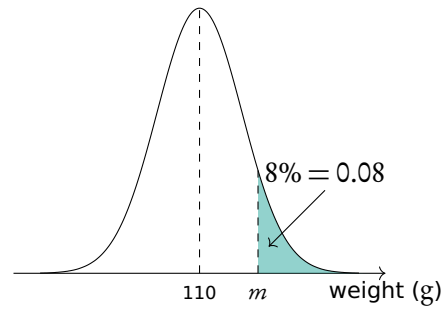
Press 


So 88.8% of the pears weigh between 100 g and 130 g.

To find an  $x$ -value when the probability is given

The weights of pears are normally distributed with mean = 110g and standard deviation = 8 g. 8% of the pears weigh more than  $m$  grams. Find  $m$ .

Sketch!



Press   
 5: Probability  
 5: Distributions  
 3: Inverse Normal

Enter probability (Area), mean ( $\mu$ ) and standard deviation ( $\sigma$ ).  
 The calculator assumes the area is to the left of the  $x$ -value you are looking for.  
 So in this case:  
 $area = 1 - 0.08 = 0.92$

Press 

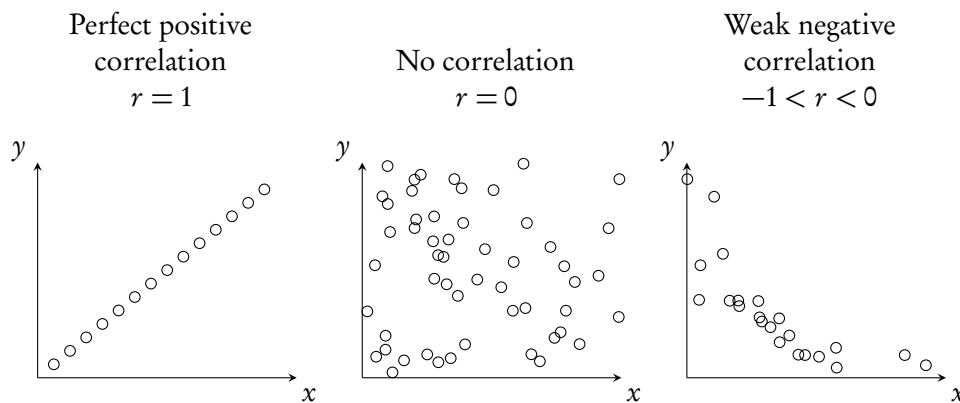
So  $m = 121$ , which means that 8% of the pears weigh more than 121 g.

## 5.2 Bivariate statistics

Bivariate statistics makes use of data where two different variables are measured. This means that you can easily plot your individual measurements as  $(x, y)$  coordinates on a scatter diagram. Analysing bivariate data allows you to assess the relationship between the two measured variables; we describe this relationship as a correlation.

The independent variable is one you have control over and the one that you expect you will have an effect on the other variable you are measuring - for instance time, age or hours of sun exposure.

### Scatter diagrams



#### 5.2.1 Pearson’s correlation: $-1 \leq r \leq 1$

Besides only estimating the correlation between two variables from a scatter diagram, you can also calculate a value that will describe it more precisely using your data. This value is referred to as Pearson’s correlation coefficient ( $r$ ).



$r = 0$  means no correlation.  
 $r \pm 1$  means a perfect positive/negative correlation.

Interpretation of  $r$ -values:

$r$ -value	$0 \leq  r  \leq 0.25$	$0.25 \leq  r  \leq 0.50$	$0.50 \leq  r  \leq 0.75$	$0.75 \leq  r  \leq 1$
correlation	very weak	weak	moderate	strong

*Remember that correlation does not mean causation.*

Calculate by finding the regression equation on your GDC: make sure STAT DIAGNOSTICS is turned ON (can be found when pressing MODE).

Bivariate statistics can also be used to predict a mathematical model that would best describe the relationship between the two measured variables; this is called regression. Here you will only have to focus on linear relationships, so only straight line graphs and equations.

Your ‘comment’ on Pearson’s correlation always has to include two things:

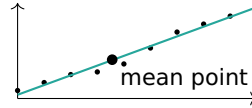
1. Positive / negative *and*
2. Strong / moderate / weak / very weak

### Find Pearson's correlation $r$ and comment on it

The height of a plant was measured the first 8 weeks

Week $x$	0	1	2	3	4	5	6	7	8
Height (cm) $y$	23.5	25	26.5	27	28.5	31.5	34.5	36	37.5

1. Plot a scatter diagram



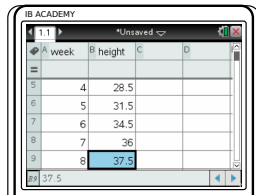
2. Use the mean point to draw a best fit line

$$\bar{x} = \frac{0 + 1 + 2 + \dots + 8}{9} = 3.56$$

$$\bar{y} = \frac{23.5 + 25 + \dots + 37.5}{9} = 30$$

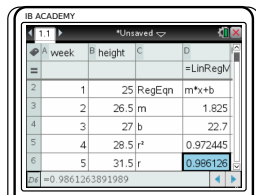
3. Find the equation of the regression line  
Using GDC

$$y = 1.83x + 22.7$$

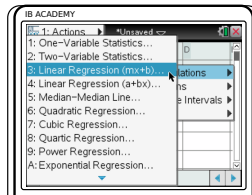


Press **off** (on), got to "Lists and Spreadsheets"

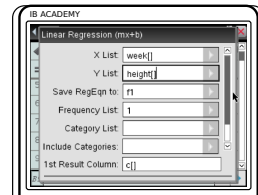
Enter  $x$ -values in one column (e.g. A) and  $y$ -values in another column (e.g. B)



So, equation of regression line is  $y = 1.83x + 22.7$  and Pearson's correlation ( $r$ -value) = 0.986



Press **menu** (menu)  
4: Statistics  
1: Stat Calculations  
3: Linear Regression (mx+b)



Enter  
X list: A [];  
Y list: B[];  
1st Result Column: C[]  
Press **OK** (OK)

4. Comment on the result.

Pearson's correlation is  $r = 0.986$ , which is a strong positive correlation.



## 5.3 Chi-square test



**Chi-square test** Used to test independence of two variables.

$H_0$  the variables are independent (null hypothesis)

$H_1$  the variables are not independent (alternative hypothesis)

### Determine if the variables are independent by the $\chi^2$ test

	Directors	Managers	Teachers	Totals
Male	26	148	448	622
Female	6	51	1051	1108
Totals	32	199	1499	1730

Perform a  $\chi^2$  test of independence at the 10% significance level to determine whether employment grade is independent of gender.

- State the null and alternative hypotheses

$H_0$ : gender and employment grade are independent  
 $H_1$ : gender and employment grade are not independent
- Calculate the table of expected frequencies

Formula:  $\frac{tf_1}{T} \cdot \frac{tf_2}{T} \cdot T$

e.g. expected number of male directors:

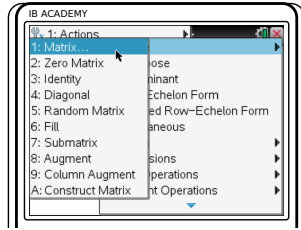
$$\frac{622}{1730} \cdot \frac{32}{1730} \cdot 1730 = 11.5$$

	Directors	Managers	Teachers
Male	11.5	71.5	539
Female	20.5	127.5	960
- Write down the degrees of freedom

$df = (\# \text{ rows} - 1)(\# \text{ columns} - 1)$

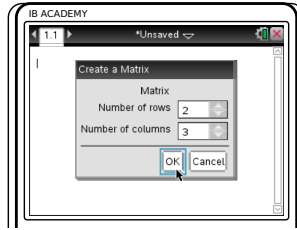
$df = (2 - 1) \cdot (3 - 1) = 2$

4. Write down the chi-square value using GDC.

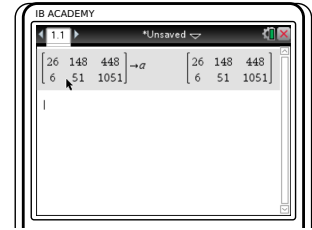


Enter data into GDC

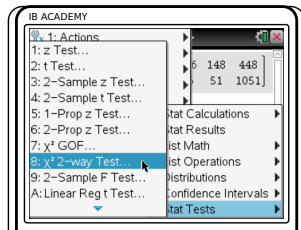
Press   
 Press   
 7: Matrices & Vectors   
 1: Create   
 1: Matrix



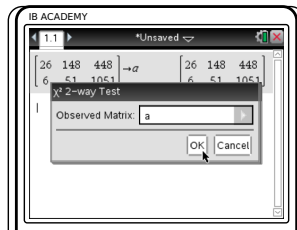
Enter dimensions of matrix to fit your data. Be sure you do *not* include the totals, so in this case you have a  $2 \times 3$  matrix   
 Press



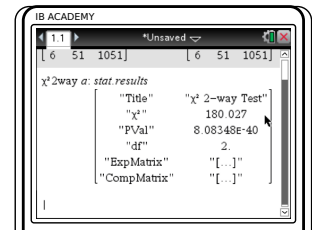
Enter the data as a matrix   
 press and   
 Give matrix a name by typing it after the arrow (e.g. a)   
 press



Press   
 6: Statistics   
 7: Statistical Tests   
 8:  $\chi^2$  2-way Test



Enter name of Observed Matrix (in this case a)   
 Press



So Chi-square value  $\chi^2 = 180.03$ , and  $p\text{-value} = 8.08 \times 10^{-40}$

# GEOMETRY AND TRIGONOMETRY

## Table of contents & cheatsheet

### 6.1. Right triangles

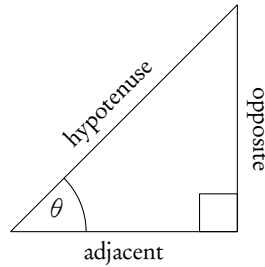
44

$$a^2 = b^2 + c^2 \quad \text{Pythagoras}$$

$$\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}} \quad \text{SOH}$$

$$\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}} \quad \text{CAH}$$

$$\tan \theta = \frac{\text{opposite}}{\text{adjacent}} \quad \text{TOA}$$



### 6.3. Surface area and volume

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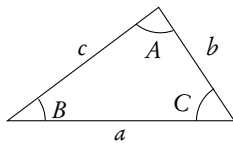
**Surface area** the sum of the areas of all faces;  
unit<sup>2</sup>

**Volume** amount of space it occupies;  
unit<sup>3</sup>

$$V = \text{area of cross-section} \times \text{height}$$

### 6.3.1. Non-right angle triangles

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Remember the angles in a triangle add up to 180°.

$$\text{Area of a triangle} = \frac{1}{2}ab \sin C$$

**Sine rule:**  $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

*Use this rule when you know:*  
2 angles and a side (not between the angles)  
or 2 sides and an angle (not between the sides).

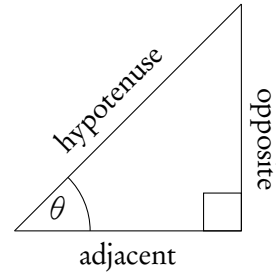
**Cosine rule:**  $c^2 = a^2 + b^2 - 2ab \cos C$

*Use this rule when you know:*  
3 sides or 2 sides and the angle between them.

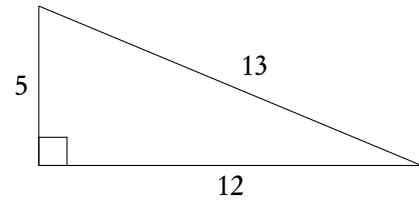
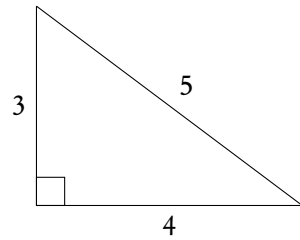
## 6.1 Right triangles



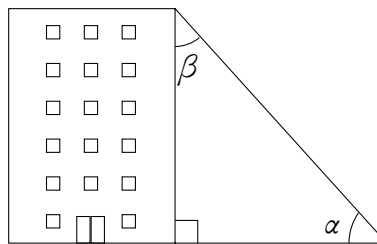
$a^2 = b^2 + c^2$	Pythagoras
$\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}}$	SOH
$\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}}$	CAH
$\tan \theta = \frac{\text{opposite}}{\text{adjacent}}$	TOA



Two important triangles to memorize:



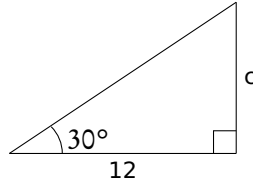
The IB loves asking questions about these special triangles which have whole numbers for all the sides of the right triangles.



$\alpha = \text{angle of elevation.}$   
 $\beta = \text{angle of depression.}$

To solve problems using Pythagoras, SOH, CAH or TOA identify what information is given and asked. Then determine which of the equations contains all three elements and solve for the unknown.

**Triangle: finding an angle or the length of a side**

 Find  $c$  in the following triangle:

**1.** Identify:

- info given
- need to find
- angle and adjacent
- opposite

**2.** pythagoras: 3x length

 SOH:  $\Theta$ , opp & hyp

 CAH:  $\Theta$ , adj & hyp

 TOA:  $\Theta$ , adj & opp

$$\tan 30^\circ = \frac{c}{12}$$

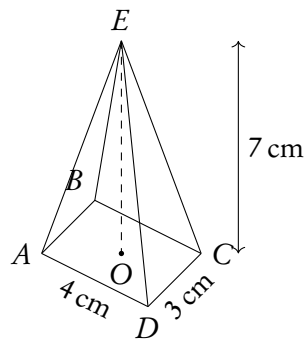
$$\Rightarrow c = 12 \times \tan 30^\circ$$

$$= 6.92$$

## 6.2 3D applications

 To find angles and the length of lines, use *SOH*, *CAH*, *TOA* and *Pythagoras*.

Example

 Rectangular-based pyramid  $ABCDE$  with  $AD = 4$  cm,  $CD = 3$  cm,  $EO = 7$  cm.

 Find the length of  $AC$ .

$$\begin{aligned} AC^2 &= AD^2 + DC^2 \\ &= 4^2 + 3^2 \\ &= 25 \\ \Rightarrow AC &= \sqrt{25} \\ &= 5 \text{ cm} \end{aligned}$$

Example.

Find the length of  $AE$ .

$$\begin{aligned} AE^2 &= AD^2 + EO^2 \\ (AO = \frac{1}{2}AC = 2.5) \\ AE^2 &= 2.5^2 + 7^2 \\ &= 55.25 \\ \Rightarrow AE &= \sqrt{55.25} \\ &= 7.43 \text{ cm} \end{aligned}$$

Find the angle  $\hat{AEC}$ .

$$\begin{aligned} \hat{AEC} &= 2\hat{AEO} \\ \tan \hat{AEO} &= \frac{2.5}{7} \\ \Rightarrow \hat{AEO} &= \tan^{-1}\left(\frac{2.5}{7}\right) \\ &= 19.65^\circ \\ \Rightarrow \hat{AEC} &= 2 \times 19.65 \\ &= 39.3^\circ \end{aligned}$$

Find the angle that  $AE$  makes with the base of the pyramid.

Looking for angle  $\hat{EAO}$ :

$$\begin{aligned} \tan \hat{EAO} &= \frac{7}{2.5} \\ \Rightarrow \hat{EAO} &= \tan^{-1}\left(\frac{7}{2.5}\right) \\ &= 70.3^\circ \end{aligned}$$

Find the angle the base makes with  $EM$ , where  $M$  is the midpoint of  $CD$ .

Looking for angle  $\hat{EMO}$ :

$$\begin{aligned} \tan \hat{EMO} &= \frac{7}{OM} \\ (OM = \frac{1}{2}AD = 2 \text{ cm}) \\ \tan \hat{EMO} &= \frac{7}{2} \\ \Rightarrow \hat{EMO} &= \tan^{-1}\left(\frac{7}{2}\right) \\ &= 74.1^\circ \end{aligned}$$

## 6.3 Surface area and volume



**Surface area** the sum of the areas of all faces  $\text{cm}^2$ .

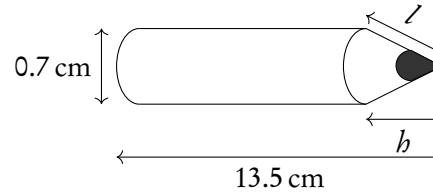
**Volume** amount of space it occupies  $\text{cm}^3$ .

$$V = \text{area of cross-section} \times \text{height}$$

Formulas in data booklet!

Example.

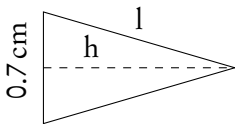
The length of the cylindrical part of a pencil is 12.3 cm



Write down the value of  $h$ .

$$\begin{aligned} h &= 13.5 - 12.3 \\ &= 1.2 \text{ cm} \end{aligned}$$

Find the value of  $l$ .



$$\begin{aligned} \Rightarrow l^2 &= \left(\frac{1}{2} \times 0.7\right)^2 + 1.2^2 \\ &= 1.5625 \\ \Rightarrow l &= 1.25 \text{ cm} \end{aligned}$$

Find the total surface area of the pencil.

$$\begin{aligned} SA_{\text{pencil}} &= SA_{\text{cylinder}} + SA_{\text{cone}} + SA_{\text{circle}} \\ &= 2\pi(0.35) \cdot 12.3 + \pi(0.35)(1.25) + \pi(0.35)^2 \\ &= 28.8 \text{ cm}^2 \end{aligned}$$

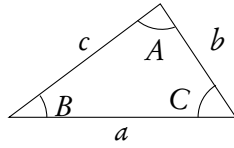
Find the volume of the pencil.

$$\begin{aligned} V_{\text{pencil}} &= V_{\text{cylinder}} + V_{\text{cone}} \\ &= \pi(0.35^2) \times 12.3 + \frac{1}{3}\pi(0.35)^2 \times 1.2 \\ &= 4.89 \text{ cm}^3 \end{aligned}$$

Read the question, does it specify if you are looking for an acute (less than  $90^\circ$ ) or obtuse (more than  $90^\circ$ ) angle. If not there may be 2 solutions. Exam hint: Use sketches when working with worded questions!

DB 3.6

### 6.3.1 Non-right angle triangles

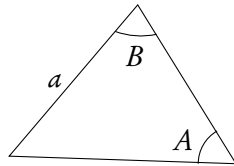


To find any missing angles or side lengths in non-right angle triangles, use the *cosine* and *sine* rule. Remember that the angles in the triangle add up to  $180^\circ$ !

**Sine rule:**  $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

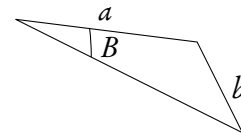
Use this rule when you know:

2 angles and a side  
(not between the angles)



or

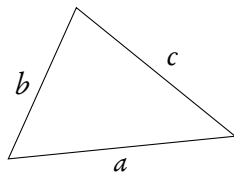
2 sides and an angle  
(not between the sides)



**Cosine rule:**  $c^2 = a^2 + b^2 - 2ab \cos C$

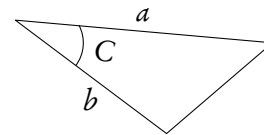
Use this rule when you know:

3 sides



or

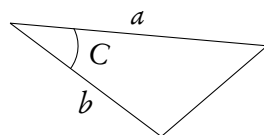
2 sides and the angle between them



**Area of a triangle:**  $\text{Area} = \frac{1}{2}ab \sin C$

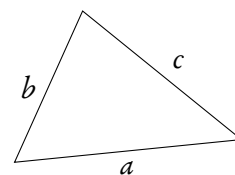
Use this rule when you know:

2 sides and the angle between them



or

3 sides  
first you need to use cosine rule  
to find an angle





Example.

$\triangle ABC : A = 40^\circ, B = 73^\circ, a = 27 \text{ cm}.$

Find  $\angle C$ .

$$\angle C = 180^\circ - 40^\circ - 73^\circ = 67^\circ$$

Find  $b$ .

$$\begin{aligned} \frac{a}{\sin A} &= \frac{b}{\sin B} \\ \frac{27}{\sin 40^\circ} &= \frac{b}{\sin 73^\circ} \\ b &= \frac{27}{\sin 40^\circ} \cdot \sin 73^\circ = 40.169 \approx 40.2 \text{ cm} \end{aligned}$$

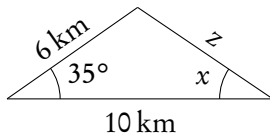
Find  $c$ .

$$\begin{aligned} \frac{c}{\sin C} &= \frac{a}{\sin A} \\ c &= \frac{27}{\sin 40^\circ} \times \sin 67^\circ = 38.7 \text{ cm} \end{aligned}$$

Find the area.

$$\begin{aligned} \text{Area} &= \frac{1}{2} \cdot 27 \cdot 40 \cdot \sin 67^\circ \\ &= 499.59 \approx 500 \text{ cm}^2 \end{aligned}$$

Example.



Find  $z$ .

$$\begin{aligned} z^2 &= 6^2 + 10^2 - 2 \cdot 6 \cdot 10 \cdot \cos 35^\circ \\ z^2 &= 37.70 \\ z &= 6.14 \text{ km} \end{aligned}$$

Find  $\angle x$ .

$$\begin{aligned} \frac{6}{\sin x} &= \frac{6.14}{\sin 35^\circ} \\ \sin x &= 0.56 \\ x &= \sin^{-1}(0.56) = 55.91^\circ \end{aligned}$$



# MATHEMATICAL MODELS

## Table of contents & cheatsheet

### Definitions

**Function** a mathematical relationship where each input has a single output. It is often written as  $f(x)$  where  $x$  is the input.

**Domain** all possible  $x$ -values that a function can have. You can also think of this as the 'input' into a mathematical model.

**Range** all possible  $y$ -values that a function can give you. You can also think of this as the 'output' of a mathematical model.

**Coordinates** uniquely determines the position of a point, given by  $(x, y)$ .

### 7.2. Linear

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$$y = mx + c$$

Where:

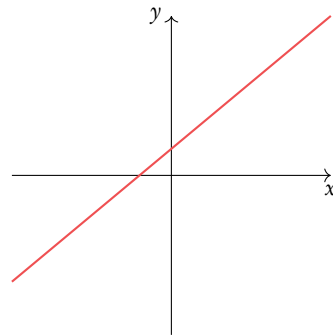
$m$  = gradient (slope)

$c$  =  $y$ -intercept

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

For parallel lines:  $m_1 = m_2$

For perpendicular lines:  $m_2 = \frac{-1}{m_1}$



### 7.3. Quadratic

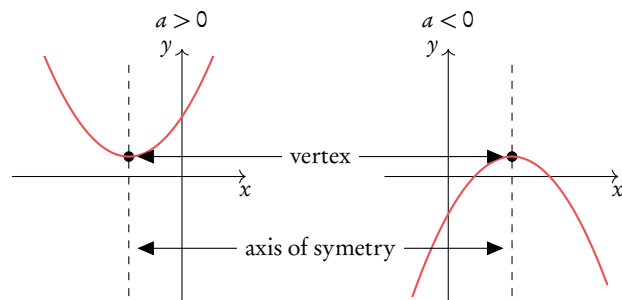
55

$$y = ax^2 + bx + c = 0$$

**Axis of symmetry**

$$x\text{-coordinate of the vertex: } x = \frac{-b}{2a}$$

**Factorized form**  $y = (x + p)(x + q)$



### 7.4. Exponential

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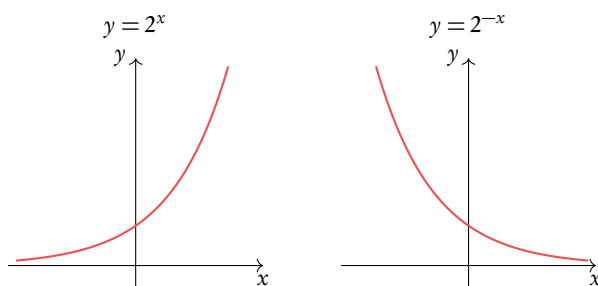
$$y = ka^x + c$$

	$a > 1$	$a < 1$
$a^x$	increasing	decreasing
$a^{-x}$	decreasing	increasing

**Horizontal asymptote**  $y = c$

**Coordinates of  $y$ -intercept**  $(0, k + c)$

**Asymptote** a line that a graph approaches but never quite touches.



## 7.1 Domain & range

Mathematical models allows you to calculate the output that a certain input will give you. To describe a mathematical model (or function) you therefore need to know the possible  $x$  and  $y$ -values that it can have; these are called the domain and the range respectively.

Note: in some questions a domain will be given to you (often even though the function as such could in theory have many other  $x$ -value inputs). This is relevant when you have to for example sketch the graph; make sure that you only draw the function for the  $x$ -values included in the domain.

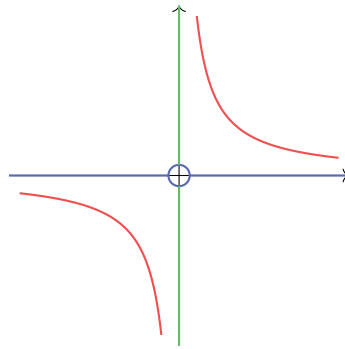


**Domain** all possible  $x$ -values that a function can have. You can also think of this as the ‘input’ into a mathematical model.

**Range** all possible  $y$ -values that a function can give you. You can also think of this as the ‘output’ of a mathematical model.

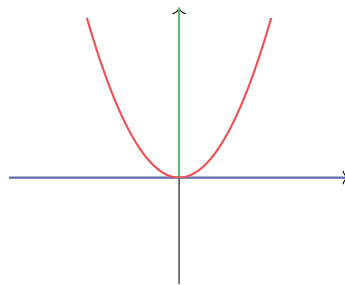
Example.

Find the domain and range for the function  $y = \frac{1}{x}$



Domain:  $x \neq 0$   
(all real numbers except 0)  
Range:  $y \neq 0$   
(all real numbers except 0)

Find the domain and range for the function  $y = x^2$



Domain:  $x \in \mathbb{R}$   
(all real numbers)  
Range:  $y \in \mathbb{R}^+$   
(all positive real numbers)

## 7.2 Linear

Linear mathematical models make straight line graphs. Two elements you need to know to describe a linear function are its slope/gradient (how steeply it is rising or decreasing) and its  $y$ -intercept (the  $y$ -value when the function crosses the  $y$ -axis, so when  $x = 0$ ).

Straight line equation is usually written in the following form:

$$y = mx + c$$

With  
 $m = \text{gradient (slope)}$   
 $c = y\text{-intercept}$

This is useful, because this way you can read off the gradient ( $m$ ) and  $y$ -intercept ( $c$ ) directly from the equation (or make a straight line equation yourself, if you know the value of the gradient and  $y$ -intercept.)

Another form in which you may see a straight line equation is:  $ax + by + c = 0$ .

In these cases, it is best to rearrange the equation into the  $y = mx + c$  form discussed above. You can do this by using the rules of algebra to make  $y$  the subject of the equation.

When you are not given the value of the gradient in a question, you can find it if you know two points that should lie on your straight line. The gradient ( $m$ ) can be calculated by substituting your two known coordinates  $(x_1, y_1)$  and  $(x_2, y_2)$  into the following equation:

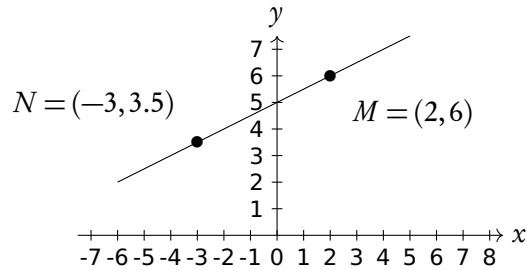
$$\frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

Make sure you substitute the  $y$  and  $x$ -coordinates in the correct order!

When you know the equation of one straight line, you can use the value of its gradient ( $m$ ) to find equations of other straight lines that are parallel or perpendicular to it.

- Parallel lines have the same slope:  $m_1 = m_2$ .
- Perpendicular lines ( $90^\circ$  angle) have:  $m_2 = \frac{-1}{m_1}$ .

**Find the equation of the line. Then find the  $x$ -intercept.**



- |    |   |  |
|----|---|--|
| 1. | Take two points on the graph and substitute the values into the formula | $m = \frac{y_2 - y_1}{x_2 - x_1}$ $= \frac{6 - 3.5}{2 - (-3)} = 0.5$ |
| 2. | Fill in one point to find $c$   | $6 = 0.5(2)$ $= 1 + c, \text{ so:}$ $c = 5$                          |
| 3. | Write down the equation $y = mx + c$ and replace $m$ and $c$            | $y = 0.5x + 5$   |
| 4. | To find the $x$ -intercept, solve by isolating $x$                      | $0 = 0.5x + 5$ $-5 = 0.5x$ $x = -10$ $\text{x-intercept: } (-10, 0)$ |

### 7.2.1 Interesection of lines

**Finding a linear equation with the gradient and a point**

Line  $L_1$  has gradient 5 and intersects line  $L_2$  at point  $A(1, 0)$ . Find the equation of  $L_1$

- |    |                               |   |
|----|-------------------------------|---|
| 1. | Find slope                    | Slope given, $m = 5$  |
| 2. | Fill in one point to find $c$ | $L_1 \text{ passes through } (1, 0)$ $\Rightarrow 0 = 5(1) + c$ $\Rightarrow c = -5$ $\Rightarrow y = 5x - 5$ |

**Finding a linear equation with perpendicular gradient and a point**

Line  $L_2$  is perpendicular to  $L_1$ . Find the equation of  $L_2$

1. Find slope

$$L_2 \text{ is perpendicular to } L_1 \text{ so}$$

$$m = -\frac{1}{\text{gradient}}$$

$$\Rightarrow m = -\frac{1}{5}$$

2. Fill in one point to find  $c$

$$0 = -\frac{1}{5}(1) + c$$

$$\Rightarrow c = \frac{1}{5}$$

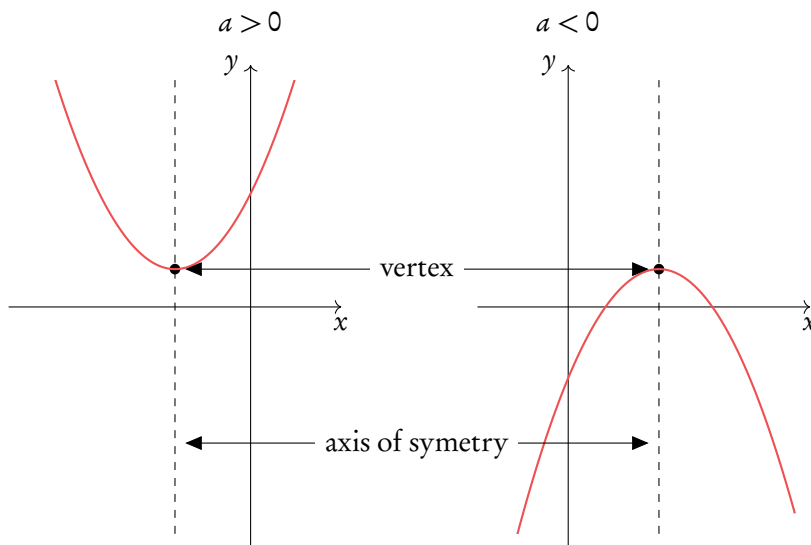
$$\Rightarrow y = -\frac{1}{5}x + \frac{1}{5}$$

### 7.3 Quadratic

Functions in which one of the terms is raised to the power of 2, written in the following form, are called quadratic:

$$y = ax^2 + bx + c = 0$$

When plotted on a graph, a quadratic function always gives an upward or downward facing U-shape – this is called a *parabola*. A parabola always has a vertex (the maximum or minimum point) and an axis of symmetry.



If you know the  $x$  and  $y$  coordinate of the vertex, the equation for the axis of symmetry will always be  $x =$  [the  $x$ -coordinate of the vertex]. This also works the other way around; the equation of the axis of symmetry gives you the  $x$ -coordinate of the vertex.

The equation for the axis of symmetry can be found using the following equation where  $a$ ,  $b$  and  $c$  are the corresponding numbers from your quadratic equation written in the form  $y = ax^2 + bx + c$ :

**Axis of symmetry:**  $x = \frac{-b}{2a} = x\text{-coordinate of vertex}$

### 7.3.1 Factorization

One more thing you have to know how to find for quadratic mathematical models are its  $x$ -intercepts. These you can find by setting your quadratic equation equal to 0. So when  $ax^2 + bx + c = 0$  you can solve for  $x$  to find the  $x$ -intercepts (or ‘roots’ as they are sometimes also called). Given that quadratic equations have the shape of a parabola, they can have up to two  $x$ -intercepts - as you can see when a quadratic equation is plotted, it often crosses the  $x$ -axis twice.

You can find the  $x$ -intercepts using several methods. Here we show an example of factorization, but you can also do this graphically using your GDC (See basics, Polyrootfinder instructions).

Example.

Factorize  $x^2 - 2x - 15$ .

$$\begin{aligned} x^2 - 2x - 15 &= (x + a)(x + b) \\ a + b &= -2 & \Rightarrow & a = -5 \\ ab &= 15 & & b = 3 \\ x^2 - 2x - 15 &= (x - 5)(x + 3) \end{aligned}$$

Find the coordinates of the  $x$ -intercepts and vertex.

$$\begin{aligned} x - 5 = 0 & & \text{or} & & x + 3 = 0 \\ x = 5 & & & & x = -3 \end{aligned}$$

$x$ -intercepts:  $(5, 0)$  and  $(-3, 0)$ .

$$\begin{aligned} \text{Vertex: } \frac{-b}{2a} &= \frac{-(-2)}{2 \cdot 1} = 1 & \leftarrow x\text{-coordinate} \\ 1^2 - 2(1) - 15 &= \\ 1 - 2 - 15 &= -16 & \leftarrow y\text{-coordinate} \end{aligned}$$

Vertex =  $(1, -16)$



## 7.4 Exponential

Another type of mathematical model that you need to be familiar with is the exponential function. An exponential function is one where the variable ( $x$ ) is the power itself. An exponential function can therefore be written in the following form:

$$y = ka^x + c$$



**Asymptote** a line that a graph approaches but never quite touches.

In questions dealing with exponential functions, you will need to know how to describe the following three things: their asymptotes,  $y$ -intercepts and whether they are increasing or decreasing. When an exponential function is written in the form described above ( $y = ka^x + c$ ), you can use its different parts to find these.

### Where is the asymptote?

The horizontal asymptote is at  $y = c$ .

### Where is the $y$ -intercept?

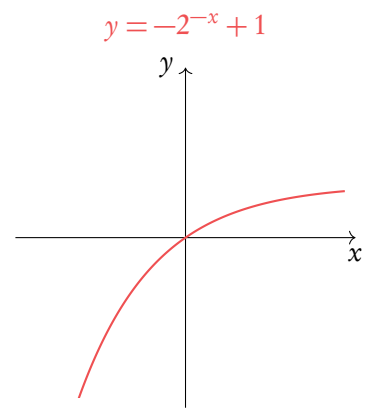
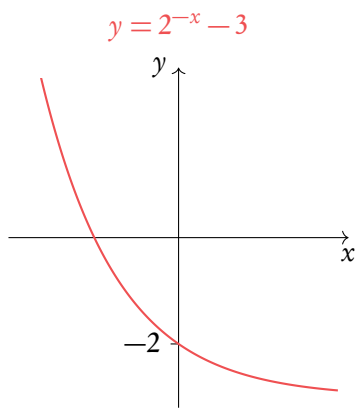
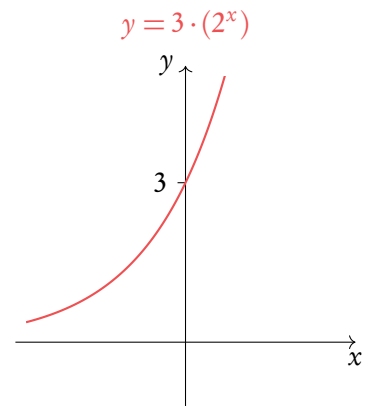
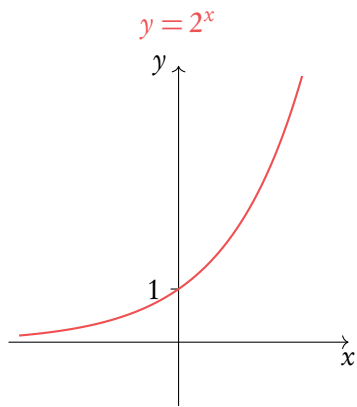
$y$ -intercept has coordinates  $(0, k + c)$ .

### Is the function increasing or decreasing?

	$a > 1$	$a < 1$
$a^x$	increasing	decreasing
$a^{-x}$	decreasing	increasing

Example.

Use GDC to sketch more complicated functions.

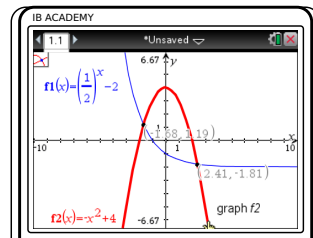
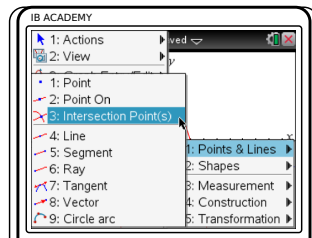
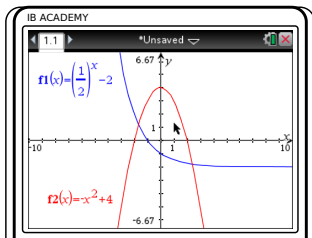


## 7.5 Intersection

When functions intersect the  $x$  and  $y$ -values are equal, so at the point of intersection  $f(x) = g(x)$ .

To find the intersection point(s) of two functions.

$$f(x) = \frac{1}{2}x - 2 \text{ and } g(x) = -x^2 + 4$$



Plot both functions

Press  $\left[ \text{on} \right]$ , go to “Graph”

Enter the two functions:

$$f1(x) = \frac{1}{2}x - 2,$$

press  $\left[ \text{tab} \right]$  to input

$$f2(x) = -x^2 + 4$$

Find the intersection:

Press  $\left[ \text{menu} \right]$

8: Geometry

1: Points & lines

3: Intersection Point(s)

Approach the intersection you are trying to find with the cursor and click once you near it. Repeat for any other intersections.

In this case the intersection points are  $(-1.68, 1.19)$  and  $(2.41, -1.81)$ .

Note: if you can't find 8: Geometry make sure you use graph mode instead of the scratchpad, otherwise please update your N-spire to the latest version.



# DIFFERENTIATION

## Table of contents & cheatsheet

### 8.1. Polynomials

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**Polynomial** a mathematical expression or function that contains several terms often raised to different powers

When  $y = f(x) = ax^n$  then the derivative is  $\frac{dy}{dx} = f'(x) = nax^{n-1}$ .

**Derivative of a constant (number)** 0

**Derivative of a sum** sum of derivatives.

When  $y = ax^n + bx^m$ ,  $\frac{dy}{dx} = na^{n-1} + mbx^{m-1}$

### 8.2. Tangent/Normal

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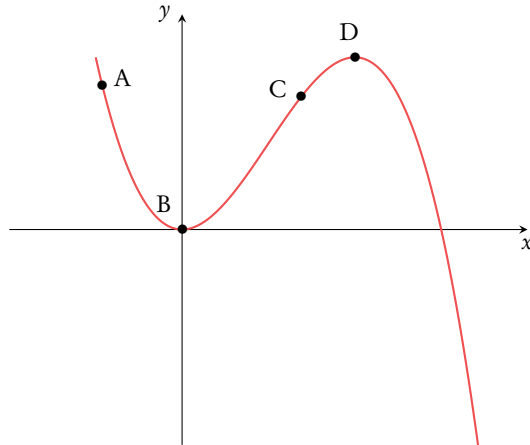
**Tangent** a straight line that touches a curve at one single point. At that point, the gradient of the curve = the gradient of the tangent.

**Normal** a straight line that is perpendicular to the tangent line. Slope of normal =  $\frac{-1}{\text{slope of tangent}}$

### 8.3. Turning points

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Turning points occur when a function has a local maximum or local minimum. At these points  $f'(x) = 0$ .



	$f'(x)$	$f(x)$ is
A	-	decreasing
B	0	at local minimum
C	+	increasing
D	0	at local maximum

## 8.1 Polynomials

As you have learnt in the section on functions, a straight line graph has a gradient. This gradient describes the rate at which the graph is changing and thanks to it we can tell how steep the line will be. In fact gradients can be found for any function - the special thing about linear functions is that their gradient is always the same (given by  $m$  in  $y = mx + c$ ). For polynomial functions the gradient is always changing. This is where calculus comes in handy; we can use differentiation to derive a function using which we can find the gradient for any value of  $x$ .

Using the following steps, you can find the derivative function ( $f'(x)$ ) for any polynomial function ( $f(x)$ ).



**Polynomial** a mathematical expression or function that contains several terms often raised to different powers

$$\text{e.g. } y = 3x^2, \quad y = 121x^5 + 7x^3 + x \quad \text{or} \quad y = 4x^{\frac{2}{3}} + 2x^{\frac{1}{3}}$$

**Principles**  $y = f(x) = ax^n \quad \Rightarrow \quad \frac{dy}{dx} = f'(x) = nax^{n-1}.$

The (original) function is described by  $y$  or  $f(x)$ , the derivative (gradient) function is described by  $\frac{dy}{dx}$  or  $f'(x)$ .

**Derivative of a constant (number)** 0

$$\text{e.g. For } f(x) = 5, \quad f'(x) = 0$$

**Derivative of a sum** sum of derivatives.

If a function you are looking to differentiate is made up of several summed parts, find the derivatives for each part separately and then add them together again.

$$\text{e.g. } f(x) = ax^n \quad \text{and} \quad g(x) = bx^m$$

$$f'(x) + g'(x) = nax^{n-1} + mbx^{m-1}$$

When differentiating it is important to rewrite the polynomial function into a form that is easy to differentiate. Practically this means that you may need to use the laws of exponents before or after differentiation to simplify the function.

For example,  $y = \frac{5}{x^3}$  seems difficult to differentiate, but using the laws of exponents we know that  $y = \frac{5}{x^3} = 5x^{-3}$ . Having the equation in this form allows you to apply the same rules again to differentiate.

Example.

$f(x)$	$f'(x)$
5	→ 0
$x^2$	→ $2 \cdot 1x^{2-1} = 2x$
$4x^3$	→ $3 \cdot 4x^{3-1} = 12x^2$
$3x^5 - 2x^2$	→ $5 \cdot 3x^{5-1} - 2 \cdot 2x^{2-1} = 15x^4 - 4x$
$\frac{2}{x^4} = 2x^{-4}$	→ $(-4) \cdot 2x^{-4-1} = -8x^{-5} = \frac{-8}{x^5}$
$3x^4 - \frac{2}{x^3} + 3$	→ $4 \cdot 3x^{4-1} - 3 \cdot (-2)x^{-3-1} + 0 = 12x^3 + \frac{6}{x^4}$

## 8.2 Tangent/Normal



**Tangent** a straight line that touches a curve at one single point. At that point, the gradient of the curve is equal to the gradient of the tangent.

**Normal** a straight line that is perpendicular to the tangent line:

$$\text{slope of normal} = \frac{-1}{\text{slope of tangent}}$$

For any questions with tangent and/or normal lines, use the steps described in the following example.

## Finding the linear function of the tangent.

Let  $f(x) = x^3$ . Find the equation of the tangent at  $x = 2$ 

- |    |  |  |
|----|--|--|
| 1. | Find the derivative and fill in value of $x$ to determine slope of tangent | $f'(x) = 3x^2$<br>$f'(2) = 3 \cdot 2^2 = 12$                   |
| 2. | Determine the $y$ value  | $f(x) = 2^3 = 8$   |
| 3. | Plug the slope $m$ and the $y$ value in $y = mx + c$                       | $8 = 12x + c$  |
| 4. | Fill in the value for $x$ to find $c$                                      | $8 = 12(2) + c$<br>$c = -16$<br>eq. of tangent: $y = 12x - 16$ |

Steps 1, 2 and 4 are identical for the equation of the tangent and normal

## Finding the linear function of the normal.

Let  $f(x) = x^3$ . Find the equation of the normal at  $x = 2$ 

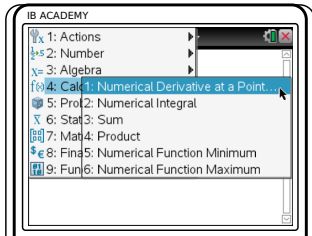
- |    |   |  |
|----|---|--|
| 1. | _____   | $f'(2) = 12$   |
| 2. | _____   | $f(x) = 8$   |
| 3. | Determine the slope of the normal<br>$m = \frac{-1}{\text{slope tangent}}$ and plug it and the $y$ -value into $y = mx + c$ | $m = \frac{-1}{12}$<br>$8 = -\frac{1}{12}x + c$  |
| 4. | Fill in the value for $x$ to find $c$   | $8 = -\frac{1}{12}(2) + c$<br>$c = \frac{49}{6}$<br>eq. of normal: $y = -\frac{1}{12}x + \frac{49}{6}$ |


Steps 1, 2 and 4 are identical for the equation of the tangent and normal

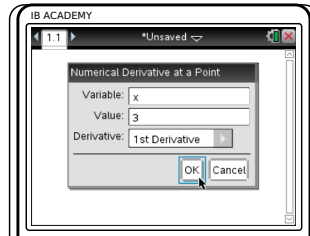



To find the gradient of a function for any value of  $x$ .

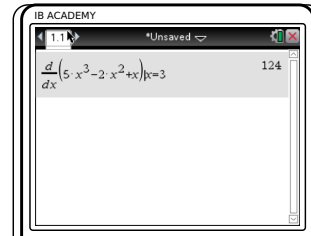
$f(x) = 5x^3 - 2x^2 + x$ . Find the gradient of  $f(x)$  at  $x = 3$ .



Press  (menu)  
4: Calculus  
1: Numerical Derivative  
at a Point



Enter the variable used in  
your function ( $x$ ) and the  
value of  $x$  that you want to  
find. Keep the settings on  
1st Derivative  
Press 



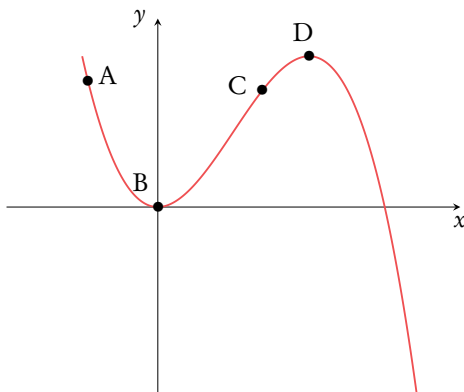
Type in your function  
press  (enter)

In this case,  $f'(3) = 124$

## 8.3 Turning points

Turning points are when a graph shows a local maximum (top) or minimum (dip). This occurs when the derivative  $f'(x) = 0$ .

Use the graph (GDC) to see whether a turning point is a maximum or minimum.



	$f'(x)$	$f(x)$ is
A	-	decreasing
B	0	at local minimum
C	+	increasing
D	0	at local maximum

## 8.4 Optimization

As we saw in the previous section, differentiation is useful for identifying maximum and minimum points of different functions. We can apply this knowledge to many real life problems in which we may seek to find maximum or minimum values; this is referred to as optimization.

Note: The most important thing to remember is that at a maximum or minimum point  $f'(x) = 0$ . So, often if a question asks a maximum/minimum value of something, like in this example, differentiation may well be a useful way to approach it.

### Determine the max/min value with certain constraints

The sum of the height  $h$  and base  $x$  of a triangle is 40 cm. Find an expression for the area in terms of  $x$ , hence find the maximum area of the triangle.

- |    |   |  |
|----|---|--|
| 1. | First write expression(s) for constraints followed by an expression for the actual calculation. Combine two expressions so that you are left with one variable. | $x + h = 40$ $h = 40 - x$ $A = \frac{1}{2}xh$ $= \frac{1}{2}x(40 - x)$ $= -\frac{1}{2}x^2 + 20x$ |
| 2. | Differentiate the expression  | $\frac{dA}{dx} = -x + 20$  |
| 3. | The derivative = 0, solve for $x$   | $-x + 20 = 0$ $x = 20$   |
| 4. | Plug the $x$ value into the original function   | $A = -\frac{1}{2}(20)^2 + 20(20)$ $= -200 + 400$ $= 200 \text{ cm}^2$                            |